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# Perturbative Approximations to Cosmologies with Nonlinear Structure

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# Declaration

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  C. Gallagher, T. Clifton, and C. Clarkson. *JCAP 03 (2020) 03, 011*

- Relativistic Euler equations in cosmologies with nonlinear structures
  C. S. Gallagher and T. Clifton. *Phys. Rev., D98(10):103516 (2018)*

- Perturbation theory for cosmologies with nonlinear structure
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# Abstract

The real universe is permeated by gravitationally collapsed structures with large density contrasts on small-scales. Cosmologists often rely on perturbative approximations that are known to be inadequate in dealing with these small-scale phenomena to model large-scale physics whilst using numerical simulations to model the nonlinear short-scales, however; due to the characteristic interplay of scales present in nonlinear systems, it is conceivable that these short-scale nonlinear structures could have an effect on long-wavelength fluctuations. The implication is that the standard practice of performing perturbation theory to calculate results on large-scales, and using N-body simulations for small-scales may be inadequate, since one may have to include the effects of the collapsed nonlinear structures in N-body simulations in the long-wavelength perturbation theory. Two-parameter perturbation theory is a perturbative scheme that has been developed specifically to deal with this problem, by allowing for short-scale nonlinear structures to be dealt with using post-Newtonian gravity whilst retaining a linear perturbation theory description of the long-wavelength universe, albeit, with the effects of short-scale nonlinear structure included. In this thesis we further develop the two-parameter perturbation theory formalism to include the two-parameter conservation equations. These equations are required to close the leading order short-scale nonlinear system in two-parameter perturbation theory. We approximate solutions to the leading-order short-scale nonlinear system by using Eulerian perturbation theory, correspondingly allowing for similar approximations to be constructed to the large-scale behaviour. These calculations are carried out in both Einstein-de Sitter and $\Lambda$CDM cosmologies. For purposes of comparison to the results of traditional perturbation theory, we calculate the two-parameter intrinsic dark matter bispectrum in both cosmologies, revealing significant differences between the two-parameter results and those from traditional relativistic perturbation theory. We discuss the viability of gauges in post-Newtonian gravity and cosmological perturbation theory, finding that only a restricted class of gauges are available for simultaneous use in both approximation schemes.

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# 1. Introduction

Humanity's understanding of the universe which it inhabits has changed beyond all recognition in the last one hundred years. Bound together by a few key physical assumptions, the theoretical machineries of particle physics, statistical mechanics and general relativity have brought forth a standard cosmological model that is able to explain a remarkable assortment of naively unconnected astronomical observations to an equally remarkable degree of accuracy [1]. When taken at face value, the standard cosmological model provides simultaneously coherent explanations for statistical variations in the imprint of the afterglow of creation on the night sky, the existence of measurable patterns in the distribution of galaxies across that very same night sky, the measured abundances of the light elements in the universe, and the observed acceleration of the expansion of the universe [2].

Despite the great successes of this model however, grave concerns remain regarding the foundational ingredients. Cosmology's best explanation for the observed late-time accelerated expansion of the universe seems to be to insert an ad-hoc constant into the field equations of general relativity, without any physical motivation to explain the origin of the constant or what it physically represents [3–6]. Initial hopes that the so-called "cosmological constant problem" might be well explained via the notion of vacuum energy in quantum field theory were quickly dashed, when theoretical calculations revealed a discrepancy of the order $10^{120}$ between the field theory prediction and the value required to fit observations [6]. Deeper investigations into the cosmological constant problem have revealed that the cosmological constant is extremely sensitive to the details of unknown high-energy physics, requiring *repeated*, *order-by-order* fine-tuning whenever higher-order loop corrections are included - known as *radiative instability*. Further questions pertain to the nature and origin of the non-luminous dark matter required to make up $\sim 95\%$ of the matter content of the universe within the standard cosmological model [7]. Finally, the precise mechanism that generates the primordial fluctuations seen imprinted in the cosmic microwave background and large-scale structure of matter on the night sky is not known [8]. This mechanism is usually expected to be provided by a theory of quantum gravity; however it has thus far proved difficult to reconcile general relativity with the quantum mechanical world of particle physics, and no experimentally testable four-dimensional theory of quantum gravity currently exists [9, 10]. Many *higher dimensional* theories of quantum gravity have been constructed in the framework of string theory [11], and other attempts at quantum gravity such as loop quantum gravity [12] and casual set theory [13] are yet to demonstrate that they contain the standard model of particle physics as a limiting case.

In contrast to the current inadequacy of theoretical modelling in cosmology, our observational capabilities have never been greater. The Planck experiment has measured the anisotropy of the cosmic microwave background to particle-physics rivalling degree of precision, improving on the results of COBE and WMAP by orders of magnitude [14–16]. Current galaxy survey experiments like the Sloan Digital Sky Survey (SDSS) [17], Kilo-Degree Survey (KIDS) [18], and the Dark Energy Survey (DES) [19] directly probe the distribution of dark matter in the universe through many different techniques including weak lensing, HI intensity mapping and galaxy number counts. Next generation galaxy surveys like Euclid [20–22], LSST [23–25] and the SKA [26] will extend observable volumes to almost the scale of the horizon. Finally, the detection of gravitational waves by Advanced LIGO in 2015 has opened an entirely new observational window through which the universe may be observed (or listened to), with tantalising prospects for cosmology [27, 28]. One hopes that advances in observational cosmology may shed light on the cause of late-time accelerated expansion, the nature of dark matter, and perhaps even the mystery of quantum gravity.

Of the ingredients in standard cosmological modelling, particle physics and statistical mechanics are well-established and tested at all accessible energy scales; however, thus far general relativity has only been subjected to stringent tests on solar system length scales [29, 30]. Given the imminence of the arrival of the next generation of galaxy survey experiments, ensuring that we precisely understand the predictions that general relativity makes on the largest scales has therefore become a matter of paramount importance. Cosmological modelling in general relativity has traditionally relied on the application of cosmological perturbation theory to an exact homogeneous and isotropic background solution, thereby enabling a description of an inhomogeneous and anisotropic universe that is in some sense close to homogeneity and isotropy [31, 32]. This methodology has been sufficient to model the anisotropy of the cosmic microwave background to a spectacular degree of precision [33].

The possibility of small-scale inhomogeneity influencing the evolution of the background solution is known as *backreaction* [34, 35], and has been studied for decades, initially in the context of looking for alternative solutions to late-time accelerated expansion. Initial research in the area was motivated by the observation that there is no consistent way to average tensors in general spacetimes, and that averaging procedures generically do not commute with the calculation of curvature tensors due to the presence of nonlinear terms. Whilst it is now accepted by the majority of authors that backreaction does not provide a satisfactory solution to the problem of late time accelerated expansion [36–38], what has not been established is whether small-scale nonlinear fluctuations could have an effect on large-scale linear perturbations.

Large-scale structure modelling is usually done by using perturbation theory to deal with fluctuations on the largest scales, whilst resorting to Newtonian N-body simulations to capture the fine detail in collapsed nonlinear structures [39]. Whilst this split approach has been predominant in the literature, it sidesteps the possibility that effects on small scales may influence physics on larger scales, or vice versa [40]. General relativity is a fundamentally nonlinear theory; the defining characteristic of nonlinear systems is the coupling of physics on different length scales (via energy cascades). As such, using perturbation theory for large-scale physics and simulations for small-scales in isolation without considering the effect that each of these might have on the other is unsatisfactory from a theoretical point of view [41]. Practically, it may be possible that predictions for large-scale structure observables made using perturbation theory (or simulations) may be biased by the failure to take the possibility of the occurrence of these effects into account. Issues like this may have a significant bearing on the interpretation of the data collected by galaxy surveys. One approach to this problem is to create general relativistic N-body simulations e.g. [42–44]. Other groups have experimented with the application of full numerical relativity to cosmology, e.g [45–49]. Neither of these approaches is limited in extent in principle, however computing power imposes limitations on the spatial extent of the models that can be constructed whilst maintain the fine-detail resolution required to accurately model nonlinear effects.

Post-Newtonian gravity is another perturbative approach to general relativity that has more traditionally found application on solar system scales [50, 51]. Unlike cosmological perturbation theory, post-Newtonian gravity is perfectly well adapted to dealing with nonlinear density contrasts on small scales; however, it is ill-suited for describing fluctuations on extremely large-scales. This is because the post-Newtonian approximation proceeds by approximating the solutions to the wave-like Einstein field equations by the solutions to Poisson-like equations, and in doing so, restricts the domain of applicability to the near-zone. This is equivalent to making a slow-motion approximation. Much recent attention has been devoted towards the use of post-Newtonian techniques in cosmological contexts [52–54].

This thesis will examine both cosmological perturbation theory and post-Newtonian gravity in the context of modelling large-scale structure, discussing the various advantages and disadvantages of each approach. The thesis will go on to present an approach to cosmological modelling that attempts to retain the positive aspects of both traditional cosmological perturbation theory, and post-Newtonian gravity, by performing both perturbative expansions simultaneously. This approach is conceptually pleasing, as each approximation scheme is used in the regime where it is expected to provide an accurate description of physical phenomena, whilst the possibility of interaction between terms present in each expansion is not neglected, as it is in standard treatments.

Chapter 2 provides a brief introduction to the mathematical underpinnings of general relativity and the standard cosmological model at the level of background evolution. A brief thermodynamic history of the universe is given, and cosmological models and problems are discussed in more detail. The mathematical machinery of the post-Newtonian approximation is developed from first principles in Chapter 3, and solutions to the post-Newtonian equations of motion are derived.

A review of the subject of cosmological perturbation theory is given in Chapter 4. We begin with a discussion of Newtonian perturbation theory, before moving on to discuss relativistic perturbation theory. Second-order relativistic perturbation theory solutions are calculated in Poisson gauge, and the tree-level bispectrum is considered in both the Newtonian and relativistic cases.

Chapter 5 discusses the viability of different gauges in both cosmological perturbation theory and post-Newtonian gravity, identifying a subset of gauges which are simultaneously viable in both approximation schemes. This result is of some importance, since one expects the nonlinear structure in the real universe to be well-modelled by the post-Newtonian approximation, but for cosmological perturbation theory to hold on large-scales. Thus if interpolation between these schemes is desirable (as it may well be if one were seeking to describe a realistic universe with nonlinear structure on small scales), one should choose a gauge which is simultaneously viable in both schemes.

Chapter 6 introduces the two-parameter expansion discussed above, reviewing its formal development, before going on to extend the formalism to include a cosmological constant and background pressure. The formalism is further developed by the derivation of the accompanying conservation equations, and a discussion of the conservation of the constraint field equations under time evolution. It is found that fluid equations are required to complete the description of the post-Newtonian equations, leading to the suggestion of utilising Newtonian perturbation theory to approximate solutions to the leading order nonlinear Newtonian field equations derived in the two-parameter expansion.

Chapter 7 deals with finding the approximate solutions to the subleading order two-parameter field equations in an Einstein-de Sitter geometry. These solutions are derived by approximating the behaviour of the fully nonlinear leading order Newtonian system using Newtonian perturbation theory, and then feeding those approximate solutions into the subleading order two-parameter field equations. The second approximation to the two-parameter field equations then displays similar behaviour to second-order relativistic perturbation theory in Poisson gauge, albeit with slightly different field equations. The evolution of the scalar metric potential is found to be identical in both cases, however differences in the structure of the constraint field equations lead to different results for the dark-matter density contrast. The effects of these differences are examined via the calculation of the tree-level bispectrum in both cosmological perturbation theory and the two-parameter expansion, with differences becoming noticeable at ultra-large-scales.

Chapter 8 extends the approximate treatment developed in Chapter 7 to the case of a universe dominated by cold dark matter and a cosmological constant. We find that the differences between cosmological perturbation theory and the two-parameter expansion are accentuated in the presence of the cosmological constant, due to the presence of a time-dependent first-order scalar metric potential (which is constant in Einstein-de Sitter geometries). This leads to differences on large-scales for the metric potentials in the two-parameter approach compared to second-order cosmological perturbation theory. These differences are visualised through the calculation of the dimensionless tree level bispectrum of the gravitational scalar potential. Differences between the dark-matter density contrast are even larger than in the Einstein-de Sitter case.

Finally, our results and conclusions are summarised in Chapter 9, alongside a discussion of further questions raised by the work presented, and ideas for future investigation.

## 1.1. Notation and Conventions

We use the metric signature $(-,+,+,+)$ and a naturalised system of units with $G = c = 1$. Our conventions for the definition of curvature tensors align with those in Misner, Thorne and Wheeler [55]. We use Greek letters $(\mu, \nu, \sigma, \dots)$ to denote spacetime indices, whilst using Latin letters $(i, j, k, \dots)$ to denote spatial indices. Partial derivatives may sometimes be denoted by a comma e.g

$$\partial_i \varphi = \frac{\partial \varphi}{\partial x^i} = \varphi_{,i} \ , \tag{1.1}$$

whilst covariant derivatives may be indicated via a semi-colon e.g.

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu{}_{\mu\lambda} v^\lambda = v^\nu{}_{;\mu} \ . \tag{1.2}$$

Einstein summation convention is in use, so any repeated indices should be understood to be summed over e.g.

$$\nabla^2 = \partial_i \partial^i = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z} \ . \tag{1.3}$$

Further notation will be introduced as needed throughout the text.

# 2. General Relativity and Cosmology

The purpose of this chapter is to motivate mathematical models of smooth homogeneous and isotropic universes around which perturbations will be developed in subsequent chapters. Towards this goal, we give short accounts of the mathematical formulation of general relativity and basic cosmology.

We will begin with a discussion of the theoretical aspects of general relativity, and define the central tools and conventions that will be used in the rest of this thesis. We assume knowledge of differential geometry; for further reference consult [56]. For a deeper discussion of general relativity, consult the textbooks by Wald, [57], Misner, Thorne and Wheeler, [55], or the first few chapters Hawking and Ellis, [58].

We will begin our consideration of cosmology with a short discussion on the nature of the Cosmological Principle. We will then progress to a discussion of the typical spacetimes used in cosmology, the Friedmann-Lemaitre-Robertson-Walker (or FLRW) class of spacetimes, restricting discussion to the spatially flat geometry used in the standard $\Lambda$CDM model. This material is well known and covered in standard cosmology textbooks. For reference, consult [59].

For completeness, we will give a short description of the thermal history of the universe, and present evidence for the occurrence of a hot big bang. This material is covered in much greater detail in the magnificent textbook by Weinberg, [60]. We will not discuss the subject of cosmic inflation at any length beyond giving a short exposition on the motivations behind its introduction, and describing the typical outcomes of inflationary models on the initial conditions for large-scale structure formation theory. A good reference for this material is the textbook by Peter and Uzan, [61].

# 2.1. General Relativity

General relativity is our best theory of gravity. In general relativity, solutions to the Einstein field equations describe the local geometric structure of spacetime, where a spacetime can be conceived of as describing an entire universe from beginning to end (if such notions are appropriate). Physical cosmology, the scientific study of *our* universe is therefore fundamentally tied to general relativity [58].

General relativity is a physical theory describing the dynamics of a *spacetime*. A spacetime can be conceptualised as the collection of all *events*, where every event in a spacetime can be labelled using local coordinates. In more plain language, any event can be labelled by when and where it happened/happens/will happen, with respect to some local observer. No geometry is *a priori* ascribed to spacetime in general relativity - instead, the geometry of a spacetime is understood to be dynamically related to its matter content. We will therefore provide a brief outline to the main geometric objects and concepts required to make a precise statement of the theory of general relativity.

## 2.1.1. Model spacetimes

In general relativity, a spacetime is modelled by a pair, $(\mathcal{M}, \boldsymbol{g})$, where $\mathcal{M}$ is a four-dimensional, $C^\infty$ smooth, connected, Hausdorff manifold called the *spacetime manifold*, comprising all events, and $\boldsymbol{g}$ is a Lorentzian metric with signature $(-,+,+,+)$ [1] defined on $T_p(\mathcal{M}) \times T_p(\mathcal{M})$, where $T_p(\mathcal{M})$ is the tangent space to $\mathcal{M}$ at any point, $p$. Together, the pair $(\mathcal{M}, \boldsymbol{g})$ satisfy the mathematical conditions required of a *pseudo-Riemannian manifold* [56].

The pair $(\mathcal{M}, \boldsymbol{g})$ are regarded as unique up to isometry. We regard two spacetimes $(\mathcal{M}, \boldsymbol{g})$ and $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ as modelling the same physical scenario if there exists a diffeomorphism

$$f : \mathcal{M} \to \tilde{\mathcal{M}} \,, \tag{2.1}$$

such that

$$\boldsymbol{g} = f^* \tilde{\boldsymbol{g}} \,, \tag{2.2}$$

where $f^*$ denotes the pullback of $f$.

[1] Sometimes the convention $(+,-,-,-)$ is used.

### 2.1.2. The metric

The metric is a bilinear, symmetric, non-degenerate mapping that takes two vectors on the tangent space at a point $p$ , $T_p(\mathcal{M})$, as its input, and returns a real number,

$$\begin{aligned}\boldsymbol{g} :\ & T(\mathcal{M})_p \times T(\mathcal{M})_p \to \mathbb{R} \ , \\ & \boldsymbol{v},\ \boldsymbol{w} \to g(\boldsymbol{v}, \boldsymbol{w}) \in \mathbb{R} \ , \\ & \text{where} \quad g(\boldsymbol{v}, \boldsymbol{w}) = g(\boldsymbol{w}, \boldsymbol{v}) \ . \end{aligned} \tag{2.3}$$

In this sense, the metric can be thought of as a generalisation of the familiar Euclidean dot product in $\mathbb{R}^n$ to the more geometrically complicated scenario of a potentially curved pseudo-Riemannian manifold.

The existence of the metric allows all non-zero vectors at a point $p$ on $\mathcal{M}$ to be be classified into three categories. For a vector $\boldsymbol{v}$, if

$$\begin{aligned} & g(\boldsymbol{v}, \boldsymbol{v}) > 0 \ , \\ & g(\boldsymbol{v}, \boldsymbol{v}) < 0 \ , \text{or}, \\ & g(\boldsymbol{v}, \boldsymbol{v}) = 0 \ , \end{aligned} \tag{2.4}$$

then $\boldsymbol{v}$ is said to be *spacelike*, *timelike*, or *null*, respectively. This situation is in stark contrast with familiar Euclidean geometry, where the scalar product of a vector with itself is always positive definite.

One can define the *components* of the metric in a particular basis by expanding the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in terms of the basis vectors:

$$g(\boldsymbol{v}, \boldsymbol{w}) = g(v^a \boldsymbol{e}_a, w^b \boldsymbol{e}_b) = g(\boldsymbol{e}_a, \boldsymbol{e}_b) v^a w^b = g_{ab} v^a w^b \ , \tag{2.5}$$

where the real numbers $g_{ab}$ are referred to as the components of the metric in the $\boldsymbol{e}_a$ basis. For the rest of this thesis, we will use the familiar local coordinate basis on $T_p(\mathcal{M})$, $\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu}$, where $x^\mu$ are scalar coordinate fields, but the reader should be aware that other bases can be chosen if they are more convenient. By considering the action of a change of coordinates $x^\mu \to y^{\tilde{\mu}}$ on the basis vectors $\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu} \to \tilde{\boldsymbol{e}}_{\tilde{\mu}} = \frac{\partial y^{\tilde{\nu}}}{\partial x^\mu} \frac{\partial}{\partial y^{\tilde{\nu}}} = \frac{\partial y^{\tilde{\nu}}}{\partial x^\mu} \tilde{\boldsymbol{e}}_{\tilde{\nu}}$ , we can easily derive expressions for the metric components in the new coordinate basis in terms of the old one:

$$g_{\tilde{\mu}\tilde{\nu}} = \frac{\partial x^\mu}{\partial y^{\tilde{\mu}}} \frac{\partial x^\nu}{\partial y^{\tilde{\nu}}} g_{\mu\nu} \ . \tag{2.6}$$

We can regard the object $\boldsymbol{g}(\boldsymbol{v}, \cdot)$ as a mapping from $T_p(\mathcal{M})$ to $\mathbb{R}$, that is to say,

$\boldsymbol{g}(\boldsymbol{v},\cdot)$ is a one-form or covector. This in turn implies that $\boldsymbol{g}$ can also be thought of as providing a bijective (one to one) mapping between $T_p(\mathcal{M})$ and $T^*(\mathcal{M})_p$, where $T^*(\mathcal{M})_p$ is the cotangent space at a point $p$. The fact that metric is nondegenerate, i.e that $\boldsymbol{g}(\boldsymbol{v},\boldsymbol{w}) \neq 0 \quad \forall \quad \boldsymbol{v},\boldsymbol{w} \neq 0$, means that this mapping is invertible. We can therefore use the metric to associate every vector $v^\mu \frac{\partial}{\partial x^\mu}$ with a corresponding one-form $v_\mu \, \mathrm{d}x^\mu$, via

$$\begin{aligned} &\boldsymbol{g} : \boldsymbol{v} \in T(\mathcal{M}) \to g(\boldsymbol{v},\cdot) \in T^*(\mathcal{M}) \ , \\ &\text{where } \ \boldsymbol{v} = v^\mu \, \partial_\mu \ , \text{ and, } \ g(\boldsymbol{v},\cdot) = g_{\mu\nu} v^\mu \, \mathrm{d}x^\nu \ = v_\nu \, \mathrm{d}x^\nu \ . \end{aligned} \tag{2.7}$$

Since the metric is an invertible mapping, one can consider the *inverse metric* :

$$\begin{aligned} \boldsymbol{g}^{-1} &: T^*(\mathcal{M})_p \to T_p(\mathcal{M}) \ , \\ \boldsymbol{g}^{-1}&\Big(\boldsymbol{g}(\boldsymbol{v},\cdot) \ , \ \cdot \ \Big) = \boldsymbol{v} \ , \end{aligned} \tag{2.8}$$

where $\boldsymbol{g}(\boldsymbol{v},\cdot)$ is the one-form associated with the vector $\boldsymbol{v}$. By considering the action of the inverse metric on two covectors in a given basis, we can derive an expression for the components of the inverse metric:

$$\begin{aligned} \boldsymbol{g}^{-1}\Big(\boldsymbol{g}(\boldsymbol{v},\cdot), \boldsymbol{g}(\cdot,\boldsymbol{w})\Big) &= \boldsymbol{g}(\boldsymbol{v},\boldsymbol{w}) \ , \\ = \boldsymbol{g}^{-1}\big(v_\mu \mathrm{d}x^\mu, w_\nu \mathrm{d}x^\nu\big) = v_\mu w_\nu \, \boldsymbol{g}^{-1}(\mathrm{d}x^\mu, \mathrm{d}x^\nu) &= v_\mu w_\nu g^{\mu\nu} = v^\mu w^\nu g_{\mu\nu} \ , \end{aligned} \tag{2.9}$$

where we have defined $\boldsymbol{g}^{-1}(\mathrm{d}x^\mu, \mathrm{d}x^\nu) = g^{\mu\nu}$ as the components of the inverse metric. But since the components of the one-forms $v_\mu$ and $w_\nu$ can be written in terms of the components of the corresponding vectors $v^\mu$ and $v^\nu$ via $v_\mu = g_{\mu\alpha} v^\alpha$ and $w_\nu = g_{\nu\beta} v^\beta$, it is easy see that in order for (2.9) to hold true, we must require

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu \ , \tag{2.10}$$

i.e that the matrix of components of the inverse metric are the matrix inverse of the matrix of components of the metric. This gives rise to the notion of "raising and lowering indices"; the metric lowers indices, and the inverse metric raises them. We will from now onwards adopt the popular physicists' abuse of notation that refers to any tensor directly by its components, i.e we will sometimes refer to $g_{\mu\nu}$ as "the metric" or the $v^\mu$ as "a vector" despite the fact that these objects are strictly only the components of the metric or a vector in a particular basis.

Considering the quantities $dx^\mu$ (distinct from $\mathrm{d}x^\mu$, the coordinate covector basis) as infinitesimal displacements between two events in a spacetime, we can construct an important invariant, *the interval*:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \ . \tag{2.11}$$

The interval, and by extension the metric, encode information about the causal structure of spacetime. If $ds^2 > 0$, two events are spacelike separated, and cannot be causally connected to one another, since no information can propagate faster than the speed of light. If $ds^2 = 0$, the two events are null separated. One event exists on the other's past lightcone, and the other event exists on the first event's future lightcone. If $ds^2 < 0$, the two events are timelike separated, and they are causally connected, existing within the interior of each other's past/future lightcones [55, 57].

### 2.1.3. Derivatives and connections

In order to do physics in our model spacetime, we require the notion of a derivative. It is of paramount importance that derivatives on manifolds be defined in a coordinate independent way. There are several different types of coordinate independent derivatives that can be consistently defined on pseudo-Riemannian manifolds. We will examine Lie derivatives and covariant derivatives, assuming knowledge of the exterior derivative on forms and exterior calculus.

#### Lie derivatives

Whilst differentiation of a function $f$ on a manifold is as easy as differentiation with respect to the coordinates in some local patch (defining a tangent vector $\partial^\mu f \, \frac{\partial}{\partial x^\mu} \in T_p(\mathcal{M})$), it is naively unclear how a vector field can be differentiated because two infinitesimally separated vectors, $\boldsymbol{X}|_p$ and $\boldsymbol{X}|_{p+\epsilon}$ live in different two different tangent spaces, $T_p(\mathcal{M})$ and $T_{p+\epsilon}(\mathcal{M})$, and therefore cannot be subtracted from one another. The *Lie derivative* allows us to differentiate one vector field $\boldsymbol{Y}$ with respect to another $\boldsymbol{X}$ to obtain a third vector field $\mathcal{L}_{\boldsymbol{X}}[\boldsymbol{Y}]$ that measures the rate of change of the vector field $\boldsymbol{Y}$ along the integral curves of vector field $\boldsymbol{X}$. The Lie derivative of a vector field at point $p$ is defined

$$\mathcal{L}_{\boldsymbol{X}}[\boldsymbol{Y}]\big|_p \equiv \lim_{\epsilon\to 0}\left(\frac{\sigma_{\boldsymbol{X}}(-\epsilon)_*\boldsymbol{Y}|_{p'} - \boldsymbol{Y}|_p}{\epsilon}\right) \in T_p(\mathcal{M}) \quad , \tag{2.12}$$

where the vector $\boldsymbol{Y}\big|_{p'}$ living in tangent space $T_{p'}(\mathcal{M})$ is pulled back to $T_p(\mathcal{M})$ using the inverse push-forward map $\sigma_{\boldsymbol{X}}(-\epsilon)_*$ defined by the flow of $\boldsymbol{X}\big|_{p'}$ [56]. It is the use of this push-forward map to connect the two tangent spaces that enables the comparison of two vector fields.

Although the Lie derivative is clearly coordinate independent, as it is defined in a manifestly coordinate independent way, it can be easily evaluated in terms of local coordinates by using the components of the vector fields in that coordinate system's coordinate basis:

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y}\big|_p = \left[X^\mu \frac{\partial}{\partial x^\mu}Y^\nu - Y^\mu \frac{\partial}{\partial x^\mu}X^\nu\right]\frac{\partial}{\partial x^\nu}\Big|_{x=x(p)} = \big[\boldsymbol{X},\boldsymbol{Y}\big]\Big|_{x=x(p)} , \tag{2.13}$$

where $\big[\boldsymbol{X},\boldsymbol{Y}\big]$ is the *Lie bracket*. The Lie bracket is a bilinear, skew-symmetric map taking two vector fields to a third vector field:

$$\begin{aligned}
\big[\,\cdot\,,\,\cdot\,\big] \ &: T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to T_p(\mathcal{M}) \ , \\
&\boldsymbol{X}\ ,\ \boldsymbol{Y} \to \big[\boldsymbol{X},\boldsymbol{Y}\big] \\
&\text{such that } \ \big[\boldsymbol{X},\boldsymbol{Y}\big] = -\big[\boldsymbol{Y},\boldsymbol{X}\big] \ , \\
&\text{and } \ \big[[\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}\big] + \big[[\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}\big] + \big[[\boldsymbol{Z},\boldsymbol{X}],\boldsymbol{Y}\big] = 0 \ ,
\end{aligned} \tag{2.14}$$

where the identity in the final line is known as the *Jacobi identity*. The Lie bracket is an extremely useful tool for understanding the symmetry properties of $\mathcal{M}$, and connects differential geometry with the theory of Lie groups and Lie algebras in a fundamental way [56].

The Lie derivative is easily generalised to action on tensors other than vector fields.

$$\mathcal{L}_{\boldsymbol{X}}[\boldsymbol{A}]\big|_p \equiv \lim_{\epsilon\to 0}\left(\frac{\sigma(\epsilon)^*\boldsymbol{A}|_{p'} - \boldsymbol{A}|_p}{\epsilon}\right) \in \mathcal{J}^q_r(\mathcal{M})\big|_p \quad , \tag{2.15}$$

where $\boldsymbol{A}\big|_p$ is a tensor of type $(q,r)$, $\mathcal{J}^q_r(\mathcal{M})\big|_p$ is the space of tensors of type $(q,r)$at point $p$, and $\sigma(\epsilon)^*$ is the pull-back diffeomorphism along the integral curves generated by $\boldsymbol{X}$. Components of the Lie derivatives of tensors in a particular coordinate basis can easily be calculated by using the components of the Lie bracket together with the requirement that the Lie derivative obey typical Leibniz rules and that the Lie derivative of a scalar reduces to the usual definition. The general result for a tensor of type $(n, m)$ is

$$\begin{aligned}
\left(\mathcal{L}_{\boldsymbol{X}}\boldsymbol{T}\right)^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} = \; & X^\alpha\partial_\alpha T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} - (\partial_\alpha X^{\mu_1})T^{\alpha\mu_2\ldots\mu_n}{}_{\nu_1\ldots\nu_m} \\
& - \cdots - (\partial_\alpha X^{\mu_n})T^{\mu_1\ldots\mu_{n-1}\alpha}{}_{\nu_1\ldots\nu_m} + (\partial_\alpha X^{\nu_1})T^{\mu_1\ldots\mu_n}{}_{\alpha\nu_2\ldots\nu_m} \\
& + \cdots + (\partial_\alpha X^{\nu_n})T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_{m-1}\alpha} \, .
\end{aligned} \tag{2.16}$$

### Connections and the covariant derivative

The Lie derivative supplies a well defined notion of differentiation with respect to a vector field; however, unfortunately it cannot be used to extend the familiar notion of a directional derivative from vector calculus in $\mathbb{R}^3$ to the manifold. This is because the Lie derivative depends not only on the value of the vector field $\boldsymbol{X}$ at point $p$, but on the values $\boldsymbol{X}$ takes in the neighbourhood around $p$ [56]. We therefore require a notion of the directional derivative for a vector field, independent from the Lie derivative. The outcome of our computation should not depend on the coordinate system chosen since the directional derivative should depend only on the intrinsic geometrical properties of $\mathcal{M}$, and it should only depend on the value the vector field $\boldsymbol{X}$ takes at point $p$.

An alternative notion of differentiation can be considered, at the cost of adding additional geometric structure into the theory. We can define an *affine connection* as a bilinear map,

$$\begin{aligned}
\nabla \; &: \mathcal{J}^1_0(\mathcal{M}) \times \mathcal{J}^1_0(\mathcal{M}) \to \mathcal{J}^1_0(\mathcal{M}) \, , \\
&\boldsymbol{X} \, , \boldsymbol{Y} \; \to \nabla_{\boldsymbol{X}}\boldsymbol{Y} \, ,
\end{aligned} \tag{2.17}$$

satisfying the following properties:

- Multiplication by a function:

$$\nabla_{f\boldsymbol{X}}\boldsymbol{Y} = f\nabla_{\boldsymbol{X}}\boldsymbol{Y} \, , \tag{2.18}$$

  which signifies that $\nabla_{f\boldsymbol{X}}\boldsymbol{Y}|_p$ is indeed a directional derivative, i.e. it only depends on the value of $\boldsymbol{X}|_p$, not the value of $\boldsymbol{X}$ at any other point [56].

- Leibniz rule:

$$\nabla_{\boldsymbol{X}}(f\boldsymbol{Y}) = \boldsymbol{X}[f] + f\nabla_{\boldsymbol{X}}\boldsymbol{Y} \, , \tag{2.19}$$

where $\boldsymbol{X}[f]$ is the natural action of the vector field $\boldsymbol{X}$ on the function $f$, which can be evaluated in a coordinate basis as $\boldsymbol{X}[f] = X^\mu \frac{\partial f}{\partial x^\mu}$.

Since the map $\nabla$ is linear, we can specify its components by considering its action on basis vectors:

$$\nabla_{\boldsymbol{e}_\mu} \boldsymbol{e}_\nu = \nabla_\mu \boldsymbol{e}_\nu = \Gamma^\lambda{}_{\nu\mu} \boldsymbol{e}_\lambda \; , \tag{2.20}$$

where we introduce the shorthand $\nabla_{\boldsymbol{e}_\mu} = \nabla_\mu$, and define the *connection coefficients*, $\Gamma^\lambda{}_{\mu\nu}$. Specifying the connection then amounts to supplying a specific choice of these connection coefficients in every chart in our atlas for $\mathcal{M}$. We can now express $\nabla_{f\boldsymbol{X}} \boldsymbol{Y}$ in terms of components of vectors in a coordinate basis, $\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu}$,

$$\nabla_{\boldsymbol{X}} \boldsymbol{Y} = X^\mu \nabla_\mu (Y^\nu \boldsymbol{e}_\nu) = X^\mu \left( \frac{\partial Y^\nu}{\partial x^\mu} + Y^\alpha \Gamma^\nu{}_{\mu\alpha} \right) \frac{\partial}{\partial x^\nu} \; . \tag{2.21}$$

By considering the action of a change of coordinates $x^\mu \to y^{\tilde{\mu}}$ on the basis vectors $\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu} \to \tilde{\boldsymbol{e}}_{\tilde{\mu}} = \frac{\partial y^{\tilde{\nu}}}{\partial x^\mu} \frac{\partial}{\partial y^{\tilde{\nu}}} = \frac{\partial y^{\tilde{\nu}}}{\partial x^\mu} \tilde{\boldsymbol{e}}_{\tilde{\nu}}$ , it is easy to derive the following well known expression for the connection coefficients $\tilde{\Gamma}^{\tilde{\lambda}}{}_{\tilde{\mu}\tilde{\nu}}$ in the coordinate basis associated to the new coordinate system $\{y^{\tilde{\mu}}\}$, in terms of $\Gamma^\lambda{}_{\mu\nu}$ :

$$\tilde{\Gamma}^{\tilde{\lambda}}{}_{\tilde{\mu}\tilde{\nu}} = \frac{\partial x^\mu}{\partial y^{\tilde{\mu}}} \frac{\partial x^\nu}{\partial y^{\tilde{\nu}}} \frac{\partial y^{\tilde{\lambda}}}{\partial x^\lambda} \Gamma^\lambda{}_{\mu\nu} + \frac{\partial y^{\tilde{\lambda}}}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial y^{\tilde{\mu}} \partial y^{\tilde{\mu}}} \; . \tag{2.22}$$

This demonstrates that the connection coefficients do not transform like the coefficients of a $(1,2)$ tensor. It can however be trivially shown that the *difference* between two connections, $\delta\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \bar{\Gamma}^\lambda{}_{\mu\nu}$ *does* transform like a $(1,2)$ tensor, using the equality of second derivatives in the second non-tensorial term.

The connection coefficients can be decomposed into symmetric and antisymmetric parts:

$$\Gamma^\lambda{}_{\mu\nu} = S^\lambda{}_{\mu\nu} + T^\lambda{}_{\mu\nu} \; , \tag{2.23}$$

where $S^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{(\mu\nu)}$ is the symmetric part of the connection and $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{[\mu\nu]}$ is the antisymmetric part, referred to in the literature as the *torsion tensor*[2].

[2]Since torsion is defined as the difference between two connections, its components must transform like a tensor. One can define torsion in terms of maps, as was done for the affine connection, but we will neglect to do so here since torsion is neglected in the standard treatment of general relativity.

An affine connection can naturally be generalised to allow action on general tensor fields, rather than just vector fields. Consider the extended map, referred to as the *covariant derivative*:

$$\begin{aligned} \nabla \ &: \mathcal{J}^1_0 \times \mathcal{J}^q_r \to \mathcal{J}^q_r \ , \\ &\boldsymbol{X} \ , \boldsymbol{T} \to \nabla_{\boldsymbol{X}} T \ . \end{aligned} \tag{2.24}$$

The covariant derivative obeys the following properties:

- Multiplication by a function:

  $$\nabla_{f\boldsymbol{X}} \boldsymbol{T} = f \nabla_{\boldsymbol{X}} \boldsymbol{T} \ , \tag{2.25}$$

  which signifies that $\nabla_{f\boldsymbol{X}} \boldsymbol{T}\big|_p$ is indeed a directional derivative, just as the connection was.

- Leibniz rule:

  $$\nabla_{\boldsymbol{X}} (\boldsymbol{T}_1 \otimes \boldsymbol{T}_2) = (\nabla_{\boldsymbol{X}} \boldsymbol{T}_1) \otimes \boldsymbol{T}_2) + \nabla_{\boldsymbol{X}} \boldsymbol{T}_1 \otimes (\nabla_{\boldsymbol{X}} \boldsymbol{T}_2) \ , \tag{2.26}$$

  where $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are two arbitrary tensor fields.

- Commutation with the contraction of indices of a tensor:

  $$\big(\nabla_{\boldsymbol{X}} \boldsymbol{T}\big)^{\dots\mu\dots}{}_{\dots\mu\dots} = \nabla_{\boldsymbol{X}} \big(\boldsymbol{T}^{\dots\mu\dots}{}_{\dots\mu\dots}\big) \ , \tag{2.27}$$

- Action on a scalar: We define:

  $$\nabla_{\boldsymbol{x}} f = \boldsymbol{X}[f] = X^\mu \partial_\mu f \ . \tag{2.28}$$

These properties ensure that the covariant derivative of a vector field reduces to the connection, and the covariant derivative of a scalar field is just the normal partial derivative. These properties are sufficient to deduce the components of the covariant derivative of a general tensor by considering contractions with known objects (the connection acting on a vector field and the covariant derivative of a scalar field).

The general result is

$$\begin{aligned}\nabla_\lambda T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = \; & \partial_\lambda T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} + \Gamma^{\mu_1}{}_{\lambda\rho} T^{\rho\mu_2\ldots\mu_m}{}_{\nu_1\ldots\nu_n} \\ & + \cdots + \Gamma^{\mu_m}{}_{\lambda\rho} T^{\mu_1\ldots\mu_{m-1}\rho}{}_{\nu_1\ldots\nu_n} - \Gamma^{\rho}{}_{\lambda\nu_1} T^{\mu_1\ldots\mu_n}{}_{\rho\nu_2\ldots\nu_n} - \ldots \\ & - \Gamma^{\rho}{}_{\lambda\nu_n} T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_{n-1}\rho} \, . \end{aligned} \tag{2.29}$$

**The Levi-Civita connection**

A connection is said to be a *metric connection* if

$$\nabla_{\boldsymbol{X}} \boldsymbol{g} = 0 \quad \forall \quad \boldsymbol{X} \;\in\; T_p(\mathcal{M}) \, , \tag{2.30}$$

at every point $p \in \mathcal{M}$ [56]. This is the requirement that the metric is parallel transported along every curve in $\mathcal{M}$. This requirement is stringent enough to directly determine a connection in terms of derivatives of the components of the metric:

$$C^{\rho}{}_{\alpha\beta} g_{\rho\mu} = \Gamma^{\rho}{}_{(\alpha\beta)} g_{\rho\mu} - \Gamma^{\rho}{}_{[\mu\alpha]} g_{\rho\alpha} - \Gamma^{\rho}{}_{[\mu\beta]} g_{\rho\alpha} \, , \tag{2.31}$$

where

$$C^{\rho}{}_{\alpha\beta} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta} \right) \, . \tag{2.32}$$

Provided we assume that $T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{[\mu\nu]} = 0$, this connection is unique, and is known as the *Levi-Civita connection*. This is the natural connection that arises in general relativity [55].

### 2.1.4. Parallel transport and geodesics

The Levi-Civita connection provides us with an intuitive definition of *parallel transport* [57]. Given a vector field $\boldsymbol{X}$ and a curve $\mathcal{C}$, parameterised by the real number $\lambda$, such that,

$$\begin{aligned} \mathcal{C} \; : \mathbb{R} \to \mathcal{M} \, , \\ \mathcal{C}(\lambda) \to x^{\mu}(\lambda) \, , \end{aligned} \tag{2.33}$$

we may consider how the vector field at a point on the curve varies. $\boldsymbol{X}$ is said to be *parallel-transported* along $\mathcal{C}$ if

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}\lambda}} \boldsymbol{X} = 0 \,. \tag{2.34}$$

In a coordinate basis, this can be expressed as

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}\lambda}} \boldsymbol{X} = \nabla_{\frac{\mathrm{d}}{\mathrm{d}\lambda}} \left( X^\mu \frac{\partial}{\partial x^\mu} \right) = \left( \frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda} + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} X^\beta \Gamma^\mu{}_{\alpha\beta} \right) \frac{\partial}{\partial x^\mu} \,. \tag{2.35}$$

Now consider the case of a curve $\mathcal{C}$ whose tangent vector at a point $p$, $\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}\lambda}|_p = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\partial}{\partial x^\mu}|_p$ is parallel-transported along itself, that is to say

$$\nabla_{\boldsymbol{v}} \boldsymbol{v}|_p = 0 \ \ \forall \ \ p \in \mathcal{C} \,. \tag{2.36}$$

Such a curve is referred to as a *geodesic* and may be thought of as being "straight" with respect to the connection, since its tangent is parallel-transported along the curve. In the coordinate basis, we require the tangent vector's components, $v^\mu$, to satisfy.

$$v^\mu \nabla_\mu v^\sigma = \frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\lambda^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \Gamma^\sigma{}_{\mu\nu} = 0 \,, \tag{2.37}$$

which is known as the *geodesic equation* (provided the connection is indeed the Levi-Civita connection), a set of four second order ordinary differential equations for $x^\mu(\lambda)$, the parametric representation of a geodesic curve. Geodesics are the natural extension of the concept of "straight" lines to intrinsically curved geometries, and are extremely important in general relativity, since light travels spacetime on null geodesics and matter travels through spacetime on timelike geodesics.

The Levi-Civita connection is the connection on a curved surface for which the curves of shortest distance are geodesics. This can be seen easily by considering functional variations of

$$l_{\mathcal{C}} = \int_{\mathcal{C}} ds = \int_{\mathcal{C}} d\lambda \sqrt{g_{\mu\nu}[x(\lambda)] \dot{x}^\mu \dot{x}^\nu} \,, \tag{2.38}$$

with respect to $x^\mu(\lambda)$, where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$. Carrying out the computation, one finds that the Euler-Lagrange equation that must be satisfied is precisely the geodesic equation, indicating that geodesics have extremal length [56].

We further take note of the fact that Equation (2.37) is only sensitive to the symmetric part of the connection, since the torsion cancels identically when contracted with the same tangent vector twice. This implies that two connections that only differ by a torsion have the same geodesics. This observation led Einstein to conclude that torsion is physically unobservable and set it identically to zero[3].

### 2.1.5. Curvature

#### Curvature map

At this point, we have all the tools required to define the *curvature map*, [56]:

$$\begin{aligned} \boldsymbol{R} \, &: \mathcal{J}^1_0 \times \mathcal{J}^1_0 \times \mathcal{J}^1_0 \to \mathcal{J}^1_0 \ , \\ &\boldsymbol{X} \ , \ \boldsymbol{Y} \ , \ \boldsymbol{Z} \ \to \boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) \ , \\ &\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} \ . \end{aligned} \tag{2.39}$$

$\boldsymbol{R}$ satisfies the following properties:

- Antisymmetry in $\boldsymbol{X}$ and $\boldsymbol{Y}$:

$$\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) = -\boldsymbol{R}(\boldsymbol{Y},\boldsymbol{X},\boldsymbol{Z}) \ . \tag{2.40}$$

- $\boldsymbol{R}$ is linear in each of its arguments.

- $\boldsymbol{R}$ satisfies:

$$\boldsymbol{R}(\,a\,\boldsymbol{X},\,b\,\boldsymbol{Y},\,c\,\boldsymbol{Z}) = \,a\,b\,c\,\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) \quad \forall \quad a,\ b,\ c \ \in \mathcal{F}(\mathcal{M}) \ , \tag{2.41}$$

  where $\mathcal{F}(\mathcal{M})$ is the space of scalar fields on $\mathcal{M}$.

#### Riemann tensor

The properties listed in the previous section imply that $\boldsymbol{R}$ defines a $(1,3)$ tensor field, the *Riemann curvature tensor*:

$$\begin{aligned} \boldsymbol{R} = R^{\alpha}{}_{\beta\mu\nu}\,\frac{\partial}{\partial x^{\alpha}} \otimes \mathrm{d}x^{\beta} \otimes \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} \ , \\ \text{where} \ \ R^{\alpha}{}_{\beta\mu\nu} = \langle \boldsymbol{e}^{\alpha}, \boldsymbol{R}(\boldsymbol{e}_{\beta},\boldsymbol{e}_{\mu},\boldsymbol{e}_{\nu})\rangle \ . \end{aligned} \tag{2.42}$$

[3]There have been attempts to construct theories involving dynamical torsion - these led to the development of Einstein-Cartan gravity and teleparallel gravity [62]

Since we are using the coordinate basis, the Lie bracket $[\boldsymbol{X}, \boldsymbol{Y}]$ vanishes, and it is easy to compute the components of the Riemann tensor in terms of the Levi-Civita connection[4].

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}{}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\rho}{}_{\mu\beta} \ . \tag{2.43}$$

The curvature tensor measures precisely how the geometry at any point $p$ on $\mathcal{M}$ deviates from $\mathbb{M}^4$, the flat Minkowski spacetime (the Lorentzian analogue of flat Euclidean space), as encoded by the failure of covariant derivatives to commute. Since directional derivatives do commute in flat geometries, one concludes that this noncommutativity is a property of *intrinsic* curvature.

The components of the curvature tensor possess a number of symmetries. Firstly, due to the antisymmetry of the curvature map, the components inherit antisymmetry on two separate pairs of indices,

$$R_{\alpha\beta\mu\nu} = R_{\beta\alpha\nu\mu} = -R_{\alpha\beta\nu\mu} \ , \tag{2.44}$$

whilst is it symmetric under the pairwise interchange $\{\alpha\beta\} \leftrightarrow \{\mu\nu\}$,

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \ . \tag{2.45}$$

$R^{\alpha}{}_{\beta\mu\nu}$ also satisfy two Bianchi identities:

- The algebraic (or "first") Bianchi identity:

$$R^{\alpha}{}_{[\beta\mu\nu]} = 0 \ . \tag{2.46}$$

- The differential ("or second") Bianchi identity:

$$R^{\alpha}{}_{\beta[\mu\nu;\lambda]} = 0 \ , \tag{2.47}$$

  where we have introduced the notation $T_{\mu\nu;\lambda} = \nabla_{\lambda}T_{\mu\nu}$ to indicate the components of a covariant derivative of a tensor.

The symmetries of the Riemann tensor reduce its number of independent components from 256 down to just 20. They correspond to precisely the second derivatives of the metric tensor that cannot be set to zero via a clever choice of coordinates.

[4]Sometimes other bases are useful, in particular orthonormal, or *tetrad* bases. In general, the Lie bracket in these bases will not vanish, so one must be careful to take it into consideration [57].

**Ricci tensor, Ricci scalar, Einstein tensor and Weyl tensor**

It is possible to define at least one contraction of the Riemann tensor without reference to the metric. The contraction

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \ , \tag{2.48}$$

yields the components of a symmetric $(0,2)$ tensor field, referred to as the *Ricci tensor*. The 10 independent components in the Ricci tensor carry information about how volumes distort under curvature [56].

When there exists a metric structure on the manifold, it is possible to define a further contraction, obtaining a scalar

$$R = g^{\mu\nu} R_{\mu\nu} \ , \tag{2.49}$$

known as the *Ricci scalar* [57]. Contracting the differential Bianchi identity, Equation (2.47), it is easy to see that we can define the components of a third, covariantly conserved tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \ , \tag{2.50}$$

known as the *Einstein tensor*.

The remainder of the information in the Riemann tensor, describing the distortion of shape under curvature, is contained within the *Weyl tensor* [57], whose components are defined by the subtraction of all possible contractions of the Riemann tensor:

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = \ & R_{\alpha\beta\gamma\delta} - \frac{1}{n-2}\left(R_{\alpha\gamma} g_{\beta\delta} + R_{\beta\delta} g_{\alpha\gamma} - R_{\alpha\delta} g_{\beta\gamma} - R_{\beta\gamma} g_{\alpha\delta}\right) \\ & + \frac{1}{(n-1)(n-2)} R \left(g_{\alpha\gamma} g_{\beta\delta} - g_{\beta\gamma} g_{\alpha\delta}\right) . \end{aligned} \tag{2.51}$$

## 2.1.6. General relativity

Having assembled the geometric tools required, we are now in a position to make a statement of the theory of general relativity. General relativity is the simplest field theory for gravity satisfying the following requirements:

- The theory should satisfy the principle of general covariance, namely that the laws of physics should be invariant under arbitrary diffeomorphisms, since coordinates do not exist *a priori* in nature.

- Special relativity is recovered (up to tidal forces) for observers in free fall.
- The spacetime metric $g_{\mu\nu}$ should reduce to the Minkowski metric $\eta_{\mu\nu}$ in the case where spacetime is flat.
- The theory should reproduce Newtonian gravity in the weak-field and slow-motion approximation.
- The field equations should be derivable from a suitable action principle.
- The field equations should contain no higher derivatives than second derivatives of the metric.
- The theory should be *local*, i.e information should propagate causally.
- The theory should be four dimensional.

The field equations are derived by varying the *Einstein-Hilbert action* with respect to fluctuations of the metric:

$$S_{EH} = \frac{1}{16\pi} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, R + S_M \; , \tag{2.52}$$

where $S_M$ is the matter action, and $g = \det(g_{\mu\nu})$ yielding

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \; , \tag{2.53}$$

the *Einstein field equations*, where $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$ is the *stress-energy tensor*, which acts as the source of spacetime curvature. The uniqueness is established by *Lovelock's theorem* as proved by David Lovelock in [63, 64].

**Theorem:**

"In 4 dimensional space, any tensor $A^{\mu\nu}$ which is

- a function only of $g_{\mu\nu}$, first derivatives of $g_{\mu\nu}$, and second derivatives of $g_{\mu\nu}$,
- linear in first derivatives of $g_{\mu\nu}$,
- symmetric,
- divergenceless,

can be written in the form

$$A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu} \ , \tag{2.54}$$

where $a$ and $b$ are constants and $G^{\mu\nu}$ is the Einstein tensor."

Lovelock's theorem was previously thought to establish that the only tensor derivable from a scalar Lagrangian density that is a function exclusively of the metric, is the Einstein tensor (with the possibility of adding a constant multiplying the metric).This in turn implied that the only possible way to modify the gravitational field equations without violating the conditions above is to add a constant term, commonly referred to as the *cosmological constant*. This theorem lead to the development of the field of *modified gravity*, wherein some or all of the above conditions are altered or relaxed [62].

Recent attempts have been made to circumvent the restrictions in Lovelock's theorem by employing a dimensional regularisation technique (more commonly used in quantum field theory) to the Einstein-Hilbert action in an arbitrary number of dimensions, and then subsequently taking the limit as $d \to 4$ [65]. This technique enables the addition of an extra term (specifically the Gauss-Bonnet term) into the Einstein-Hilbert action. The full extent of the modifications to gravity that result from this theory are as of yet unclear, however as of 2020, it is an area of intensive research [66].

By virtue of the differential Bianchi identity, the stress-energy tensor obeys the *stress-energy conservation equations*,

$$\nabla_\mu T^\mu{}_\nu = 0 \ , \tag{2.55}$$

which extend the usual continuity and fluid equations from the flat case to curved spacetime. It is worth nothing that these equations are implied by the Einstein field equations and are not independent - they are simply a consequence of the Bianchi identity. The stress-energy conservation equation can however be independently derived in a curved spacetime using Noether's theorem, and thus does have an independent physical meaning from the field equations - in fact it was precisely the fact that the stress-energy tensor is covariantly conserved that motivated Einstein to look for field equations of the form of Equation (2.53). In the case of a perfect fluid, the stress energy tensor takes the form

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu} \ , \tag{2.56}$$

where $\rho$ is the energy density of the fluid, $P$ is the pressure, and $u^\mu$ is the 4-velocity vector field of the fluid.

It is worth making a few points about the Einstein equations:

- They are a set of 10 hyperbolic nonlinear second order partial differential equations for the components of the metric.
- Whilst the equations are nonlinear, they are linear in second derivatives.
- Whilst there are 10 equations for 10 degrees of freedom, the contracted differential Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, yields four more constraint equations, reducing the number of degrees of freedom down to 6. The four fictitious degrees of freedom are really gauge-fixing degrees of freedom, representing the freedom to choose coordinates in which to carry out calculations.
- The system admits a well-posed initial value problem; suitable initial data given on a spacelike Cauchy surface possesses a unique future evolution.

It is also worth taking note of the fact that although we like to describe the stress-energy tensor as "the source of curvature", we are not free to specify arbitrary stress energy tensors pertaining to arbitrary distributions of matter, in the way that one specifies charge and current distributions in electromagnetism. The stress energy tensor will, in general, depend on the metric, as in Equation (2.56). Accordingly, we can only really look at stress-energy tensors in the context of specific metrics, investigating whether solutions exist. Of course, one could then select an arbitrary metric, calculate the $G_{\mu\nu}$ and then claim the existence of a stress-energy tensor taking precisely the form that $G_{\mu\nu}$ took, and that therefore one had found a solution. This, however, fails to say anything meaningful or physical; one must make certain demands of $T_{\mu\nu}$. We require $T_{\mu\nu}$ to represent "realistic" sources of energy and momentum [58]. The usual requirement, called the *weak energy condition* states

$$T_{\mu\nu} u^\mu u^\nu \geq 0 \quad \forall \quad \text{timelike } u^\mu,\ u^\nu \ , \tag{2.57}$$

which is equivalent to the requirement that no negative energy densities are allowed in the theory.

Taking Lovelock's theorem into consideration, the only possible way to modify the Einstein-Hilbert action by adding a constant term, the so-called *cosmological constant*, $\Lambda$, such that:

$$S_{EH} = \frac{1}{16\pi} \int_{\mathcal{M}} \mathrm{d}^4x \, \sqrt{-g} \left( R - 2\Lambda \right) + S_M \ , \tag{2.58}$$

in which case the field equations become

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \ . \tag{2.59}$$

Whilst the cosmological constant was initially discarded by Einstein, who famously referred to it as "the greatest mistake of his life", it has subsequently found use in the celebrated ΛCDM model of cosmology, the details of which we will discuss in the next section.

### 2.1.7. Spacetime symmetries and Killing vectors

Let $(\mathcal{M}, \boldsymbol{g})$ be a spacetime and let $\xi^\mu$ be a smooth vector field on $\mathcal{M}$. An integral curve of $\xi^\mu$ is a curve $x^\mu(\lambda)$ such that

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}|_p = \xi^\mu|_{p'} \ . \tag{2.60}$$

If $\xi^\mu \neq 0$ then the set of integral curves is a *congruence*, i.e. every point in $\mathcal{M}$ is passed through by exactly one curve. The integral curves can then be used to generate a one-parameter family of diffeomorphisms,

$$h_s(p) \ : \ \mathcal{M} \to \mathcal{M} \ , \tag{2.61}$$

where $h_s(p)$ is a point parameter distance $s$, away from $p$ along an integral curve through $p$. This family of diffeomorphisms is referred to as a *flow*. As, defined earlier, the Lie derivative of a tensor with respect to $\xi^\mu$ measures the rate of change of that tensor under the integral flow of $\xi^\mu$. Accordingly, if $\boldsymbol{\mathcal{L}_\xi T} = 0$, we say that the tensor $\boldsymbol{T}$ has a *symmetry*, generated by $\xi^\mu$. If

$$\boldsymbol{\mathcal{L}_\xi}\, \boldsymbol{g} = 0 \ , \tag{2.62}$$

then the spacetime has a symmetry generated by $\xi^\mu$. $\xi^\mu$ is then referred to as a *Killing vector field* (or just Killing vector). It can be shown that making the replacement $\partial \to \nabla$ in Equation (2.16) leaves the result unaffected. Then, since $\nabla_\alpha g_{\mu\nu} = 0$ as we have the Levi-Civita connection, Equation (2.62) can be written in coordinates as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \ , \tag{2.63}$$

which is known as *Killing's equation* [56, 57].

If $g_{\mu\nu}$ expressed in coordinates $\{x^1, \ldots, x^n\}$ is independent of the coordinate, $x^1$, i.e if $\frac{\partial}{\partial x^1} g_{\mu\nu} = 0$, then the vector $\xi^\mu = (1, 0, \ldots, 0)$ is a Killing vector. The converse statement is also true, namely that if $\xi^\mu$ is a Killing vector, then we are able to find a set of local coordinates in which the components of $\xi^\mu$ are $\xi^\mu = (1, 0, \ldots, 0)$.

## 2.2. Cosmology

We have thus far assembled the mathematical machinery required to make a model universe. Our model universe is a spacetime $(\mathcal{M}, g_{\mu\nu})$, which has the mathematical structure of a psuedo-Riemannian manifold, equipped with the Levi-Civita connection. The dynamical behaviour of spacetime is given by a solution to the Einstein field equation, (2.53) and the associated stress energy tensor [58, 67].

### 2.2.1. Cosmological Principles

Of course, directly solving equation (2.53) for an arbitrary distribution of matter and energy is completely unfeasible due to the nonlinearity of the field equations, notwithstanding the fact that knowledge of such a distribution could never realistically be obtained [55]. Accordingly, we make simplifying assumptions, usually by *a priori* specifying the geometry and stress-energy tensor have a symmetry [67]. In cosmology, the simplest and most common symmetry assumed goes by the name of the *Cosmological Principle*, and can be stated in its simplest form as:

> *The universe is spatially homogeneous and isotropic.*

This version of the cosmological principle is, of course, to be understood in this context as a very rough-and-ready approximation. The universe is very obviously not homogeneous and isotropic, at least on the length scales observable with the naked human eye. A good analogy is perhaps that of a crystal lattice - the lattice is evidently inhomogeneous on the scale of individual atoms and electrons, however on length scales greater than each group of basis atoms, the repeating structure is homogeneous [68]. If one were to consider *coarse-graining* or *smoothing* on length scales greater than the the length scale of the lattice, one would recover a perfectly homogeneous distribution. This analogy should not be taken too seriously - obviously we are not suggesting that the universe has a lattice structure, but in the context of some hypothetical coarse-graining over a suitable length scale, we should expect to recover a homogeneous universe. At this stage we will content ourselves with a *statistical* interpretation of this principle and the knowledge that the principle is only assumed to hold true on extremely large length scales (larger than $\sim 100\,\mathrm{Mpc}$), scales larger than the largest known gravitationally bound structures. If we want to ask and answer questions about the universe without worrying about fine detail, the cosmological principle will allow us to simplify the Einstein equations considerably [55, 67].

What observational evidence is there to support such a drastic assumption? The best piece of observational evidence is the proximity to isotropy of the temperature of *cosmic microwave background* radiation (CMB radiation). The Planck Satellite has now measured the amplitude of the deviations from isotropy to be of the order of one part in $10^5$. The near-isotropy of CMB radiation alone however only provides evidence for isotropy. In order to extend isotropy to homogeneity, one would additionally have to introduce the *Copernican Principle*, namely the assumption that no observer occupies a special place in the universe. Isotropy in conjunction with the Copernican Principle implies the Cosmological Principle. Large-scale structure simulations like *gevolution* and the Millenium simulation provide another key piece of evidence for the Cosmological Principle. In order to obtain results that agree with observations from these simulations, it is necessary to assume the Cosmological Principle [69, 70]. Further observational evidence for large scale anisotropy consistent with the amount measured from the CMB has been found in galaxy clusters, quasars and radiogalaxies. For further discussions of cosmological principles, see references [67, 68]

### 2.2.2. FLRW models

How to implement the cosmological principle mathematically? We will restrict ourselves to consideration of a perfectly homogeneous and isotropic universe, permeated by a single perfect fluid. This leads us to consideration of the FLRW class of spacetimes. With spatial homogeneity and isotropy imposed, the metric takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -a^2(\tau)d\tau^2 + a^2(\tau)\left\{\frac{dr^2}{1-kr^2} + r^2 d\,\Omega^2\right\}, \tag{2.64}$$

where $\tau$ is conformal time, defined by $dt = a\,d\tau$ $d\,\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, and $a(\tau)$ is referred to as the *scale factor* and measures the expansion of the universe. $k$ is the *curvature* parameter, $k < 0$ corresponds to a hyperbolic geometry, $k = 0$ to a flat geometry and $k > 0$ to a spherical geometry. Universes with these geometries are referred to as *open*, *flat* and *closed* [59]. For the remainder of this thesis, we shall restrict ourselves to consideration of flat ($k = 0$) universes, due to their consistency with observational data [14, 59][5].

How should we model the matter in our universe, subject to the constraint that the Cosmological Principle is obeyed? Firstly, we should take note of the fact that the fluid approximation is expected to hold well if the *mean free path* of the constituent particles is significantly less than the length scales of physical interest. Secondly, the Cosmological Principle explicitly forbids the presence of anisotropic pressure. Accordingly, perfect fluid models are commonly used in cosmology [60]. We should, however, take note of the limitations of the fluid approximation. Even in thermal equilibrium, the fluid approximation begins to break down below the *Baryon Acoustic Oscillation scale*, or *BAO scale*, which is approximately $\sim 150$Mpc today. For more information on baryon acoustic oscillations, see [72].

Given the stress energy tensor for a single perfect fluid, the Einstein equations then take the form;

$$\mathcal{H}^2 = \frac{8\pi a^2}{3}\rho \, , \tag{2.65}$$

$$\mathcal{H}' = -\frac{4\pi a^2}{3}\big(\rho + 3P\big) \, , \tag{2.66}$$

where $\mathcal{H} = \frac{a'}{a}$ is the *conformal Hubble parameter* and dashes indicated differentiation with respect to conformal time. Equation (2.65) is referred to as the *Friedmann equation* and Equation (2.66) is referred to as the *Acceleration equation* or *Raychaudhuri equation*. These equations also imply

$$\rho' + 3\mathcal{H}(\rho + P) = 0 \, , \tag{2.67}$$

the *continuity equation*, which can also be derived independently via $\nabla_\mu T^\mu{}_\nu = 0$. A final equation is required to close the system,

$$P = w\rho \, , \tag{2.68}$$

which is referred to as the *equation of state*, where $w$ is a constant in the simplest approximations, but can vary in more complex models.

If we include the cosmological constant in the Einstein-Hilbert action, the previous

[5] Some recent studies have suggested that non-zero curvature models may be a better fit to the Planck data, referring to the current tensions as evidence for a “crisis in cosmology”, but this is still considered a minority viewpoint [71]. This issue was also analysed in the original Planck papers [14].

equations are simply modified to read

$$\mathcal{H}^2 = \frac{8\pi a^2}{3}\rho + \frac{1}{3}\Lambda a^2 \ , \tag{2.69}$$

$$\mathcal{H}' = -\frac{4\pi a^2}{3}\left(\rho + 3P\right) + \frac{1}{3}\Lambda a^2 \ . \tag{2.70}$$

In cosmology, any non-relativistic matter is referred to as *dust*, and is considered to have negligible pressure. This approximation is well justified since the pressure of a non-relativistic ideal gas is given by $P = nk_BT = \frac{\rho}{m}k_BT$, where $k_B$ is the Boltzmann constant, $T$ is the temperature and $m$ is the rest mass of the particles forming the gas. The ratio $\frac{k_BT}{m}$ is typically extremely small when considering galaxy mass, thereby justifying a pressure-free approximation for galaxies and their associated dark matter halos. Such matter is sometimes also referred to as *cold matter* [60]. This form of matter is modelled by the simple equation of state, $w = 0$. The energy density of cold matter can be shown to evolve as

$$\rho_m(\tau) \propto \frac{\rho_m(\tau_0)}{a^3} \ , \tag{2.71}$$

using the continuity equation. For the case of a spatially flat, cold matter-dominated (the "Einstein-de Sitter" or "EdS") universe, i.e $\rho \gg P \implies w = 0$, the system then has a simple solution,

$$\frac{a(\tau)}{a(\tau = \tau_0)} = \frac{\tau^2}{\tau_0^2} \ , \tag{2.72}$$

where $\tau_0$ is the value of the conformal time coordinate at the present time. It is convenient to choose a normalisation such that $a(\tau_0) = \tau_0 = 1$, then we have the simple relation $a = \tau^2$ for matter dominated universes.

Highly relativistic (sometimes called "hot") perfect fluids of matter and radiation are both modelled by equations of state for which $w = \frac{1}{3}$. The evolution of the density of radiation can be shown using the continuity equation to evolve as

$$\rho_r(\tau) \propto \frac{\rho_r(\tau_0)}{a^4} \ , \tag{2.73}$$

and in the case of a radiation dominated universe, the solution for the scale factor takes the form

$$\frac{a(\tau)}{a(\tau = \tau_0)} = \frac{\tau}{\tau_0} = \tau \ , \tag{2.74}$$

given the previous normalisation.

The final form of "matter" that is usually considered in cosmology is not really matter at all. Due to its ambiguous nature one is free to consider the cosmological constant as either a correction to the geometry of spacetime, or as representing an additional matter contribution in the stress energy tensor, generated by some hitherto unknown form of hypothetical matter (often referred to in the literature as "dark energy") [73]. Taking the latter perspective, and defining

$$T^{(\Lambda)}_{\mu\nu} = \frac{\Lambda}{8\pi} g_{\mu\nu} \ , \tag{2.75}$$

$$\rho_\Lambda = \frac{\Lambda}{8\pi} \ , \tag{2.76}$$

it is easy to see that in order for the continuity equation

$$\rho'_\Lambda + 3\mathcal{H}(\rho_\Lambda + w\rho_\Lambda) = 0 \tag{2.77}$$

to be satisfied consistently, we require an equation of state of the form $w = -1$ to describe the effects of the cosmological constant. The naive implication of this equation of state seems to be that if one were to *decrease* the density of the fluid, the pressure would *increase* - no observed form of matter has this property. Nevertheless, we shall see that $\Lambda$ has its place in the standard model of cosmology, although we hope to someday provide a physical explanation for its peculiar effects. For more information on dark energy, see the comprehensive textbook by Amendola and Tsujikawa, *Dark Energy*, [73]. In the case of a positive-$\Lambda$-dominated universe, the Friedmann equation reduces to

$$\mathcal{H}^2 = \frac{\Lambda a^2}{3} \ . \tag{2.78}$$

Changing back from conformal time to proper time, we find

$$H^2 = \frac{\Lambda}{3} \ , \tag{2.79}$$

where $H = \dot{a}/a$ is the Hubble parameter and $\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t}$. This equation then has the simple solution

$$a = a_0\, e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)} \ . \tag{2.80}$$

This is known as the *de Sitter* solution; it describes the dynamics of a universe whose expansion rate is accelerating exponentially.

It will be useful to define the following quantity, known as the *critical density*

$$\rho_c = \frac{3H^2}{8\pi} = \frac{3\mathcal{H}^2}{8\pi a^2} \,, \tag{2.81}$$

which corresponds to the density at present time required for the universe to have a flat geometry. It is a matter of convention in cosmology that the following *density parameters* are defined with respect to this critical density

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{8\pi a^2 \rho_m}{3\mathcal{H}^2} = \frac{\mathcal{H}_0 \Omega_{m0}}{a\mathcal{H}^2} \,, \tag{2.82}$$

$$\Omega_r = \frac{\rho_r}{\rho_c} = \frac{8\pi a^2 \rho_r}{3\mathcal{H}^2} = \frac{\mathcal{H}_0 \Omega_{r0}}{a^2\mathcal{H}^2} \,, \tag{2.83}$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{a^2 \Lambda}{3\mathcal{H}^2} = \frac{\mathcal{H}_0 \Omega_{\Lambda 0}}{\mathcal{H}^2} \,, \tag{2.84}$$

where the subscript "0" indicates that the quantity in question has been evaluated at the present value of conformal time, $\tau = \tau_0$. Using these definitions, the Friedmann equation for a general mixture of matter satisfying $\rho = \rho_m + \rho_r + \rho_\Lambda$ becomes

$$\mathcal{H}^2 = \mathcal{H}_0^2 \left( \frac{\Omega_{m0}}{a} + \frac{\Omega_{r0}}{a^2} + \Omega_{\Lambda 0} a^2 \right) \,, \tag{2.85}$$

or in coordinate time,

$$H^2 = H_0^2 \left( \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \Omega_{\Lambda 0} \right) \,, \tag{2.86}$$

and the evolution of the cosmic scale factor can be determined given measurements of $\mathcal{H}_0$ and any two of $\Omega_{m0}$, $\Omega_{r0}$ and $\Omega_{\Lambda 0}$, since for flat universes

$$\Omega_m + \Omega_r + \Omega_\Lambda = 1 \,, \tag{2.87}$$

for all time. The Planck satellite has measured that $\Omega_{r0} \sim 10^{-4}$ [14]. Accordingly, we neglect this term in studies of the late universe, and define the ΛCDM class of models as those satisfying

$$\Omega_m + \Omega_\Lambda = 1 \,. \tag{2.88}$$

Given the Planck value (assuming a six parameter flat ΛCDM cosmology [14]),

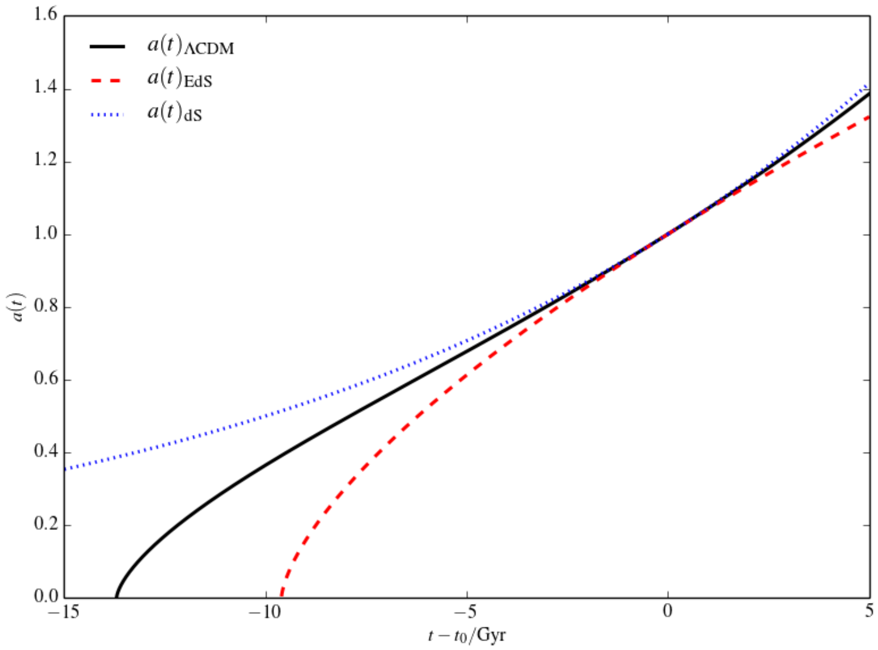


Figure 2.1.: A plot of the time evolution of the scale factors in ΛCDM, EdS and de Sitter solutions. Coordinate time is used as opposed to conformal time, in order to give the reader an intuitive sense of the age of the universe.

$\Omega_{m0} = 0.315 \pm 0.013$, we can then solve the Friedmann equation to obtain the behaviour of the scale factor. No closed form solution exists in conformal time, however in coordinate time, we have the remarkable analytic solution

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{t}{t_\Lambda}\right) , \tag{2.89}$$

where $t_\Lambda = 2/(3H_0\sqrt{\Omega_\Lambda})$ [60].

In ΛCDM models, the density in the present day universe $\Omega_{m0}$ can be subdivided into baryonic matter and cold dark matter, both of which scale like dust. We will discuss this issue more when we come to discuss the hot big bang and and the thermal history of the universe.

### 2.2.3. The Big Bang

It can be seen from Figure 2.1 that in the EdS model, the scale factor $a_{\mathrm{EdS}} \to 0$ in a finite amount of coordinate time, $t_0 - t_{\mathrm{BB}} \sim 9.5\,\mathrm{Gyrs}$, where $t_0$ is the present time, such that $a(t_0) = 1$ and $t_{\mathrm{BB}}$ is the time of the singularity, for which $a(t_{\mathrm{BB}}) = 0$. Since the energy density of both cold matter and radiation scale as inverse powers of the scale factor, it is clear that they diverge at $t_{\mathrm{BB}}$. This moment, $t_{\mathrm{BB}}$ is referred

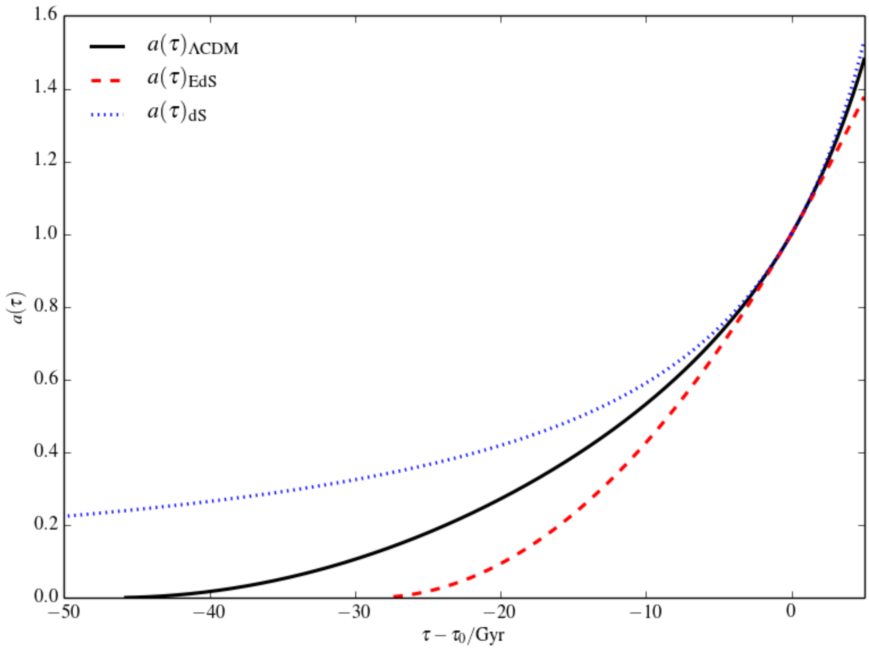


Figure 2.2.: A plot of the time evolution of the scale factors in ΛCDM, EdS and de Sitter solutions using the conformal time variable instead of coordinate time.

to as a *Big Bang singularity* - such singularities are generic features of single fluid Friedmann models with $-\frac{1}{3} < w < 1$ [59]. Let us examine why this is the case.

Consider a fluid with equation of state

$$P_w = w\rho_w \tag{2.90}$$

and define the associated density parameter

$$\Omega_{w0} = \frac{\rho_{w0}}{\rho_{c0}} \; . \tag{2.91}$$

The continuity equation can be directly integrated to obtain

$$\frac{\rho_w}{\rho_{w0}} = \left(\frac{a_0}{a}\right)^{3(1+w)} = \text{const.} \; . \tag{2.92}$$

Considering the acceleration equation written in coordinate time

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho_w + 3P_w) \; , \tag{2.93}$$

we can see that if $\dot{a} > 0$ at time $t_0$, then a fluid satisfying $-\frac{1}{3} < w < 1$ will have

$\ddot{a} < 0$, and $a(t)$ will necessarily be a concave function. Therefore, there must exist a singularity at finite coordinate time in the past.

By contrast, in the de Sitter model, where $w = -1$, there is no big bang, just an exponential decay in the scale factor as we travel back in time.

Whilst the argument given above only holds true in the case of a single fluid model, it can be seen that in the case of the ΛCDM model, a big bang singularity also occurs at finite coordinate time. The fact that ΛCDM models provide such a good fit to all known cosmological observations indicates that one should take the proposal that the universe began with a big bang extremely seriously.

What then is the nature of a big bang singularity? Firstly, the big bang singularity is a classical curvature singularity, as predicted by general relativity [58]. This is established by calculation of the *Kretschmann scalar*,

$$K = R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} \ , \tag{2.94}$$

which for flat FLRW spacetimes is given by

$$K = 12 \, \frac{a^2 \, \ddot{a} + \dot{a}^4}{a^4} \ . \tag{2.95}$$

This quantity diverges as $a \to 0$, signifying that the singularity is a true curvature singularity, as opposed to an artefact of the coordinate system we have chosen to use for our calculation.

How should we understand the occurrence of such a divergence in our model of the universe? One could take the viewpoint that general relativity is the correct theory of gravity to use on all scales, and interpret this as a genuine prediction of the initial state of the universe being a gravitational singularity. However a more popular interpretation is slightly more agnostic, namely that one expects the typical effects of *quantum gravity* to become dominant around the Planck scale, and that when we look far enough back into the past to reach those energy scales, we would have to update the predictions of general relativity to those of some theory of quantum gravity. In this case we understand the occurrence of the singularity as merely signifying our ignorance of the true physics going on at the Planck scale, and revise our claim to merely the statement that the universe was extremely hot, dense and compressed approximately 14 Gyrs ago, and has subsequently expanded according the predictions of general relativity.

### 2.2.4. Thermal history

A variety of cosmological observations have constrained the value of the density parameter of cold matter at the current time to be

$$\Omega_{m0} = 0.315 \pm 0.007 \ . \tag{2.96}$$

This cold matter can be subdivided into *baryonic matter*, namely stars and gas, but also non-baryonic dark matter

$$\Omega_{m0} = \Omega_{b0} + \Omega_{c0} \ , \tag{2.97}$$

$$\Omega_{b0}h^2 = 0.0224 \pm 0.0001 \ , \tag{2.98}$$

$$\Omega_{c0}h^2 = 0.120 \pm 0.001 \ , \tag{2.99}$$

where $\Omega_{b0}$ is the baryon fraction and $\Omega_{c0}$ is the fraction of cold non-baryonic dark matter according to the latest measurements by the Planck collaboration [14, 74], and $H_0 = 67.4 \pm 0.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = 100\,h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Non-baryonic dark matter is implied to exist by a number of independent cosmological and astrophysical observations. In particular, Big Bang nucleosynthesis places extremely strong constraints on the baryon fraction, requiring the rest of the measured matter fraction to be comprised of some non-luminous form of matter [60]. The radiation component of the density is given by

$$\Omega_{r0}\, h^2 = (\Omega_{\gamma 0} + \Omega_{\nu 0})\, h^2 = 4.15 \times 10^{-5} \ , \tag{2.100}$$

$$\Omega_{\gamma 0}\, h^2 = 2.47 \times 10^{-5} \ , \tag{2.101}$$

$$\Omega_{\nu 0}\, h^2 = 1.68 \times 10^{-5} \ , \tag{2.102}$$

where $\Omega_{\gamma 0}$ is the density parameter for the energy density in the CMB photons and $\Omega_{\nu 0}$ is the density parameter for the energy density in the cosmic neutrino background. $\Omega_{\gamma 0}$ can be directly related to the CMB temperature, corresponding to a black-body spectrum with temperature, $T_{\gamma 0} = 2.725 \pm 0.001$. The value for the energy density of neutrinos is given under the standard model assumption of the existence of three massless neutrinos, although substantial evidence has now accumulated indicating that neutrinos do in fact have a small mass [75]. The values for these parameters are taken from [76], however they are well known throughout the literature.

Using these values, and the scaling of each of the different contributions, it is possible to trace back the thermal history of the universe. Combining this with the standard model of particle physics and statistical mechanics, we can quantitively understand the different epochs at which different forms of matter dominated the cosmological dynamics. It is this cosmological timeline, constructed via a combination of general relativity, particle physics and statistical mechanics, that is technically referred to as the *hot big bang model*, and which provides our best understanding of the history of the universe, excluding any interval of time where quantum gravity effects may become important [60, 61].

We will give a short account of the cosmological timeline, in order to place the main focus of this work in its proper context. For more details, see Chapter 3 of Weinberg's textbook, *Cosmology*, [60].

### Cosmological timeline

- $t_{\mathrm{BB}} = 0 < t < 10^{-25}\,\mathrm{s}$   Speculative physics:

  Nothing concrete is known about this epoch, except that the energy scales are high enough (the Planck scale is approximately $10^{19}\,\mathrm{GeV}$) that the effects of quantum gravity become important. Cosmologists have good reason to believe that the universe underwent a period of accelerated expansion at some point towards the end of this period, commonly referred to as *inflation*. We expect inflation to occur around the energy scale of $10^{16}\,\mathrm{GeV}$ or less, otherwise one might expect to measure primordial gravitational waves. After inflation ends, the universe must undergo a process known as *reheating*, where the energy density in the hypothetical inflaton field is transferred back into the known standard model fields. The evolution of the universe then changes from being described by de Sitter expansion to radiation dominated expansion.

- $t \sim 10^{-25}\,\mathrm{s}$   Baryogenesis:

  At this scale, quantum gravity effects are no longer expected to be important, but it is possible that the effects of grand unified theories of particle physics may still be important. At some point, an unknown reaction must have occurred to generate the matter-antimatter asymmetry we observe in the real universe. Theoretical models for this process are still speculative.

- $t \sim 10^{-10}\,\mathrm{s}$ Electroweak scale:

  At this scale, the temperature is low enough for the strong force to have separated from the electroweak interaction, but electroweak symmetry breaking has not yet occurred. Physics at this scale is very well understood compared to previous epochs.

- $t \sim 10^{-4}\,\mathrm{s}$ Quark-Hadron phase transition:

  A QCD phase transition from quark-gluon plasma to bound hadron states is predicted to have occurred at this stage in the universe's evolution.

- $t \sim 10\,\mathrm{s}$ Neutrinos decouple:

  Prior to this era, neutrinos were in thermal equilibrium with protons and electrons, due to the weak interaction. When the rate of weak interactions becomes significantly lower than the rate of expansion of the universe, the weak interaction is no longer able to maintain thermal equilibrium, and neutrinos decouple, subsequently *free streaming* without scattering until the present epoch. This process generates the so-called *cosmic neutrino background*, analogous to the cosmic microwave background.

- $t \sim 10^{2}\,\mathrm{s}$ Big Bang nucleosynthesis:

  As the universe cools and the typical energy of a photon becomes smaller than the binding energies of the light elements, photons will no longer be able to dissociate nuclei that have formed, and bound nuclei will become commonplace in the universe. The physics of this process is extremely well understood, and is sufficient to precisely predict the abundances of the light elements in the universe today, using only the density of baryonic matter and the number of massless neutrino species in the universe as input parameters. Impressively, the observed abundances of light elements can only be matched with observations if precisely three species of massless neutrinos exist, corresponding to the predictions of the standard model of particle physics. This in turn provides an extremely stringent constraint on the density of baryonic matter in the universe, $0.016 < \Omega_b\, h^2 < 0.024$. The discrepancy between this number and the total matter density, $\Omega_{m0} = 0.315$, provides some of the strongest evidence for the existence of nonbaryonic dark matter.

- $t \sim 10\,\text{kyrs}$ Matter comes to dominate over radiation:

Since the density of radiation decreases as $a^{-4}$, but the density of matter only decreases as $a^{-3}$, eventually matter comes to dominate over radiation. During matter dominated expansion small fluctuations in the matter density are prone to grow in magnitude.

- $t \sim 300\,\text{kyrs}$ Photons decouple, CMB formed:

When the universe has cooled to the extent that the typical energy of a photon is no longer large enough to ionise light nuclei, the nuclei will recombine in a process called *recombination.* After recombination, the universe will no longer contain large amounts of ionised plasma, causing the typical interaction length scale of photons to become extremely large. At this point, photons *decouple* from the rest of the material (much like neutrinos before them) and travel along geodesics until we observe them in the form of the CMB today.

- $t \sim 0.1\,\text{Gyrs}$ Large-scale structures begin to form:

Small overdensities in the matter density that have been increasing in amplitude during the era of matter domination have now gained enough mass to undergo gravitational collapse. First stars are formed.

- $t \sim 5\,\text{Gyrs}$ Dark energy comes to dominate over matter

The matter density in the universe has diluted enough by this point for the effects of dark energy to begin to manifest. The expansion of the universe begins to accelerate again. Modelling of dark energy is highly speculative, since none of the known standard model fields can produce accelerated expansion.

- $t \sim 10\,\text{Gyrs}$ Life begins to evolve:

Earth is thought to have been formed over 4.5 Gyrs ago. The first single celled organisms are thought to have appeared just over a billion years after the formation of Earth.

- $t \sim 13.8\,\mathrm{Gyrs}$ Present-day epoch:

  Collapsed nonlinear structures now permeate the universe. Complex multicellular life exists on Earth.

### 2.2.5. Initial conditions and cosmological inflation

A number of unanswered questions and problems posed by the hot big bang model remain. Problems like the nature of dark matter, dark energy, or the question of matter-antimatter asymmetry are outside the scope of this work. We will however give a short qualitative account of cosmological *inflation*, a simple extension of the hot big bang model that resolves a number of problems, namely the *horizon problem*, the *flatness problem* and the problem of *magnetic monopoles*. Inflation posits that there was a period of exponential expansion of the scale factor in the very early universe, before even baryogenesis. This period of exponential expansion is usually hypothesised to have been caused by the physical effects of some hitherto unknown and unmeasured particle, referred to as the *inflaton*. We will not give details of any of the large number of specific models of inflation, instead focusing on the reasons for its suggestion, and its effects on the theory of large-scale structure. For more details, see Chapters 4 and 10 of [60] or the textbook by Peter and Uzan, *Primordial Cosmology*, [61].

#### Why inflation?

The *horizon problem* refers to a logical paradox that occurs due to the finite age of the universe in big bang models. When one considers two patches of the night sky that are too far away from each other to have been in causal contact at any point in the history of the universe in the standard $\Lambda$CDM picture, there is no mechanism that would cause these two regions to thermalize. However, the observed isotropy of the CMB suggests that the radiation fluid was indeed thermal equilibrium at all points in the observable universe at the time of decoupling.

A period of rapid accelerated expansion before baryogenesis resolves this problem very simply - during the period of accelerated expansion, the particle horizon expands exponentially, causing a region larger than the size of the observable universe today to have been in causal contact at much earlier times.

The *flatness problem* is another example of unphysical fine tuning. In a matter or radiation dominated model, any deviation from flatness tends to grow over time; positively curved universes become more positively curved, and negatively curved universes become more negatively curved. The observation that the density of the universe today is extremely close to the critical density therefore implies either that an unknown physical mechanism that sets the geometry of the universe exactly to flat, or that some fine-tuning occurred such that the density of the universe in the past was *even closer* to the critical value than it is now.

The *monopole problem* refers to the observation that generic Grand Unified Theories (GUTs) from particle physics predict the efficient creation of heavy relic particles at energy scales accessible in the early universe. The fact that these particles, the most famous of which are *magnetic monopoles*, are yet to be observed then seems highly improbable, and their absence requires explanation.

Inflation solves both the flatness problem and the monopole problem in a similar way. A period of exponential expansion before baryogenesis would dilute the density of monopoles such that the probability of measuring one becomes extremely small. Similarly, the effect of accelerated expansion is to drive the curvature to zero, flattening the universe.

### Large-scale structure

The fact that an early period of accelerated expansion solves issues with the hot big bang model is not the only reason why cosmologists find inflation to be such a compelling theory. Aside from providing convenient explanations for the issues listed above, cosmological inflation gives rise to a wonderful and intuitive explanation for the emergence of *large-scale structure* in the late universe [68].

Inflation connects inhomogeneities (deviations away from a perfect Friedmann model) in the late universe to tiny *quantum fluctuations* that occurred during the inflationary period. These tiny quantum fluctuations are then stretched out to superhorizon scales by the inflationary process. One can then consider the gravitational evolution of a universe that consists of a smooth FLRW background plus small perturbations entirely using the mechanics of general relativity after inflation. Since gravity is only attractive, the general picture is one in which overdense regions of the universe will continue to accrete more and more matter as the evolution process goes on. Fluctuations initially as small as one part in $10^{-9}$ will then grow and grow, eventually reaching a critical point at which gravitational collapse will occur, eventually forming gravitationally bound objects like galaxy clusters. In this way, the near homogeneity of the universe is reconciled with the presence of inhomogeneous large-scale structure, with the quantum fluctuations generated during inflation providing the initial conditions for the formation of structure in the universe [77]. Further information on the connection between the quantum fluctuations of inflation and large-scale structure can be found in the textbook *Cosmological Inflation and Large-Scale-Structure* by Liddle and Lyth, [78].

A mathematical description of this procedure will be given in the subsequent chapter, where we discuss perturbative approximations to general relativity and their use in cosmology.

### 2.2.6. Alternatives to the standard model

It should be noted that the basic scenario described in this chapter is not the only paradigm in which cosmological models are constructed, merely the most successful and popular. Modifications can be made to almost every step in the chain of logic, from altering the theory of gravity entirely [62], to abandoning the cosmological principle (e.g. inhomogeneous/anisotropic cosmological modelling, or altering the inflationary paradigm (e.g. the Mixmaster universe - see [55]). In general however, alternatives have struggled to reproduce the accuracy of the predictions for the anisotropy of the cosmic microwave background and the abundances of light elements in the universe without resorting to adding additional freedom into models, often suffer from fundamental pathologies (e.g. Ostrogradsky instabilities or "ghosts"), and simple extensions of $\Lambda$CDM are disfavoured [14, 79].

# 3. Post Newtonian Gravity

In this chapter, we will introduce the notion of a weak-field expansion, and develop the theoretical background necessary to consider post-Newtonian expansions. This theoretical material is essential for understanding the formal development of two-parameter perturbation theory. Material from this chapter is related to that presented in the paper [80]. Derivations of higher order post-Newtonian stress energy conservation equations on a Friedmann background were carried out by Timothy Clifton and myself.

## 3.1. Weak-field expansions

In the previous chapter, we presented a qualitative explanation for the existence of large-scale structure in the universe; namely that small fluctuations away from perfect homogeneity and isotropy are stretched out in wavelength by inflation, and then evolve under gravity, eventually collapsing into the clusters and superclusters we observe today.

It is clear that to describe such a physical scenario mathematically, we will have to move beyond the restrictive assumptions of total homogeneity and isotropy that lead to the FLRW metric. The mathematical tool we will use to describe a universe with inhomogeneities is the *weak-field expansion.* The central assumption of any weak-field assumption is that one spacetime with perturbations can be described in some sense as "close" to another without them. That is to say, we can decompose a *slightly inhomogeneous* spacetime into an exact solution and some remainder,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \ , \tag{3.1}$$

where $g_{\mu\nu}$ is the metric describing the full spacetime (i.e the spacetime with the inhomogeneous perturbations, $(\mathcal{M}, g_{\mu\nu})$), $\bar{g}_{\mu\nu}$ is an exact solution of the field equations, belonging to the spacetime $(\bar{\mathcal{M}}, \bar{g}_{\mu\nu})$, and $\delta g_{\mu\nu}$ represents deviations away from the exact solution, such that $\delta g_{\mu\nu} \ll \bar{g}_{\mu\nu}$, for some suitably chosen background.

Obviously, the choice of which exact solution $\bar{g}_{\mu\nu}$ to perturb will have a large bearing on the subsequent development of techniques for approximating $\delta g_{\mu\nu}$, as will other considerations, such as the typical behaviour and character of matter we wish to model, and the scales of interest in the problem. The constraint that $\delta g_{\mu\nu} \ll \bar{g}_{\mu\nu}$ is small is locally obeyed almost everywhere in the universe except in the near vicinity of neutron stars or black holes (the typical strong gravity astrophysical scenarios considered in the literature).

## 3.2. Post-Newtonian gravity

Post-Newtonian gravity is an example of a weak-field approximation to general relativity satisfying an additional *slow motion condition*,

$$v_c \ll 1 \; , \tag{3.2}$$

that is to say, characteristic velocities in the system under consideration must be small relative to the speed of light.

### 3.2.1. Landau-Lifshitz formulation of GR

The first step in constructing a post-Newtonian approximation is the observation that the Einstein Field Equations admit an exact reformulation as a wave equation, using the *Landau-Lifshitz formulation* of general relativity. First, we define the tensor densities,

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \; , \tag{3.3}$$

$$H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \; . \tag{3.4}$$

The tensor density $H^{\alpha\mu\beta\nu}$ shares the symmetries of the Riemann tensor and satisfies the following identity

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 2(-g)G^{\alpha\beta} + 16\pi(-g)t^{\alpha\beta}_{LL} \; , \tag{3.5}$$

where $t^{\alpha\beta}_{LL}$ is an object known as the *Landau-Lifshitz pseudotensor* that can be constructed from products of $g^{\mu\nu}$, $\mathfrak{g}^{\mu\nu}$ and derivatives of these quantities, and admits a (loose) interpretation as an energy-momentum pseudotensor for the gravitational field[1]. The normal Einstein Field Equations then imply that

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 16\pi(-g)T^{\alpha\beta} + 16\pi(-g)t^{\alpha\beta}_{LL} , \tag{3.6}$$

whilst the antisymmetry of $H^{\alpha\mu\beta\nu}$ on its last pair of indices imply

$$\partial_\beta \partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 0 , \tag{3.7}$$

$$\partial_\beta \big[(-g)\big(T^{\alpha\beta} + t^{\alpha\beta}_{LL}\big)\big] = 0 \ . \tag{3.8}$$

In particular, the last equation inspires the interpretation of $t^{\alpha\beta}_{LL}$ as being related to the energy-momentum of the gravitational field, and these equations together are equivalent to the standard expression, $\nabla_\mu T^{\mu\nu} = 0$.

The preceding equations form the basis of the Landau-Lifshitz formulation of general relativity. The reformulation is exact; however, we will not derive the equations since the calculation is extremely lengthy. We refer the interested reader to the textbook *The Classical Theory of Fields* by Landau and Lifshitz for more details [81]. The Landau-Lifshitz formulation is primarily useful in situations where the coordinates are, in some sense, close to Lorentzian coordinates $x^\mu = (t, x^i)$, and the gothic inverse metric, $\mathfrak{g}^{\mu\nu}$ deviates only slightly from the Minkowski metric, $\eta^{\mu\nu}$.

### 3.2.2. Relaxed Einstein equations

We can introduce the following *harmonic coordinate conditions*:

$$\partial_\mu \, \mathfrak{g}^{\mu\nu} = 0 , \tag{3.9}$$

along with the potentials

$$h^{\mu\nu} = \eta^{\mu\nu} - \mathfrak{g}^{\mu\nu} \ . \tag{3.10}$$

The harmonic coordinate conditions are then equivalent to

$$\partial_\mu h^{\mu\nu} = 0 , \tag{3.11}$$

[1]This interpretation is not to be taken literally - $t^{\mu\nu}_{LL}$ can always be made to vanish at any event in spacetime by a choice of Riemann normal coordinates in the neighbourhood of that specific event.

which is often referred to as the *harmonic gauge condition*, whilst the left hand side of the Einstein equation becomes

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = -\Box h^{\alpha\beta} + h^{\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta} - \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} \ , \tag{3.12}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the regular d'Alembertian in flat space. We can then write the formal wave equation

$$\Box h^{\alpha\beta} = -16\pi\tau^{\alpha\beta} \ , \tag{3.13}$$

known as the *relaxed Einstein equation* where

$$\tau^{\alpha\beta} = (-g)\big(T^{\alpha\beta}[m,g] + t^{\alpha\beta}_{LL}[h] + t^{\alpha\beta}_{H}[h]\big), \tag{3.14}$$

is the effective energy-momentum pseudotensor, and where

$$(-g)t^{\alpha\beta}_{H} = \frac{1}{16\pi}\left(\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta}\right) \ , \tag{3.15}$$

is an additional harmonic gauge contribution to the effective energy-momentum pseudotensor called the *harmonic pseudotensor*. The square brackets indicate that $T^{\alpha\beta}$ is a functional of both the matter variables and the metric, whilst the Landau-Lifshitz and harmonic pseudotensors are functionals only of the potentials $g^{\mu\nu}$. It can be shown that the harmonic gauge condition enforces the conservation of the effective energy-momentum pseudotensor,

$$\partial_\mu \tau^{\mu\nu} = 0 \ . \tag{3.16}$$

The relaxed Einstein equations, together with the harmonic gauge condition constitute an exact reformulation of general relativity. Provided the potentials satisfy the harmonic gauge condition (or the effective energy-momentum tensor is conserved) and the relaxed field equation, it is guaranteed that $g^{\mu\nu}$ will be a solution of the Einstein equation. The relaxed field equation determines the behaviour of the metric in terms of the matter variables, whilst the conservation equation (3.16) determines the behaviour of the matter variables in a curved spacetime described by the potentials $h^{\mu\nu}$. One is not free to solve the full Einstein equation for the metric independently of the matter variables; however, it is possible to integrate the relaxed field equation for the potentials independently of imposing the gauge condition via the conservation equation. It is true though, that the solutions for the potentials will only correspond to a solution of the full Einstein equation once the conservation equation is subsequently imposed.

The relaxed Einstein equation possesses a formal solution in terms of a *retarded Green's function*;

$$h^{\alpha\beta}(x) = 4\int G(x,x')\tau^{\alpha\beta}(x')\,\mathrm{d}^4x' \ , \tag{3.17}$$

where $x = (t, \boldsymbol{x})$ is the point at which we evaluate the field, and $x' = (t, \boldsymbol{x}')$ is a source point. $G(x, x')$, which is evaluated at both points is the retarded Green's function, which satisfies

$$\Box G(x,x') = -4\pi\delta^{(4)}(x-x') \ . \tag{3.18}$$

It can easily be verified by differentiation and a quick integration by parts that the conservation equation $\partial_\beta \tau^{\alpha\beta} = 0$ guarantees that this solution will satisfy the harmonic gauge condition.

### Iterative approximations and post-Minkowskian series

Although we have written a formal solution, it is still unclear how to proceed if explicit expressions are required, since $\tau^{\alpha\beta}$ is implicitly dependent on $h^{\alpha\beta}$. The usual procedure followed in post-Newtonian gravity is to construct an asymptotic series approximation around the Minkowski metric, referred to as a *post-Minkowskian approximation*[2], of the form

$$h^{\alpha\beta} = Gk_1^{\alpha\beta} + G^2k_2^{\alpha\beta} + G^3k_3^{\alpha\beta} + \ldots \ = \ h_0^{\alpha\beta} + h_1^{\alpha\beta} + h_2^{\alpha\beta} + \ldots \ , \tag{3.19}$$

where $h_0^{\alpha\beta} = 0$, $h_1^{\alpha\beta} = Gk_1^{\alpha\beta}$ etc. The relaxed field equation is then solved using an iterative method, wherein the source function at a particular order can be constructed from quantities already solved for at a previous order. In this way, the problem of the implicit dependence of the source on the potentials is alleviated at each order, and the problem is reduced to solving the wave equation for a known source function at each order. The final step in constructing such an approximation is to impose the harmonic gauge condition on the final result. We will not go over the precise mathematics required to evaluate this iterative construction, since our purpose is to construct a similar type of expansion around an FLRW metric. We

[2] The use of $G$, a quantity with dimensions as an expansion parameter is to be regarded as a formal device - the actual dimensionless expansion parameter depends on the characteristic mass and length scales of the system being modelled.

will ultimately choose a different approach, where an expression similar to the form of Equation (3.19) will be substituted into the Einstein equation, and then terms with the same power in the expansion parameter are equated. Although this is not usually done in post-Newtonian theory, it is important to note that it could be done in principle - the iterative treatment is just more popular in the literature. A standard iterative treatment of post-Minkowskian theory is given in the textbook *Gravity* by Poisson and Will [51].

### 3.2.3. Post-Newtonian expansions around FLRW

Let us write the wave equation symbolically (ignoring tensor indices for ease) as

$$\Box\psi = -4\pi\mu \;, \tag{3.20}$$

where $\psi$ is a generic quantity standing in for $h_n^{\alpha\beta}(x)$ and $\mu$ represents the (known) source function.

The retarded Green's function for the wave equation for a known source function is given by:

$$G(x, x') = \frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} \;. \tag{3.21}$$

A derivation can be found in any standard textbook on electromagnetism, for reference we recommend pages 183-185 of [82]. We can substitute this specific form for the Green's function into the formal solution,

$$\psi(x) = \int G(x, x')\mu(x')\, \mathrm{d}^4x' \tag{3.22}$$

and integrate over $t'$ to write the retarded solution to the wave equation as

$$\psi(t, \boldsymbol{x}) = \int_{\mathcal{C}} \frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}^3x' \;, \tag{3.23}$$

where domain of integration $\mathcal{C}(x)$ extends over the *past light cone* of the point at which we are evaluating $\psi(\boldsymbol{x}, t)$. The retarded solution represents a superposition of null waves.

The domain $\mathcal{C}(x)$ is generally partitioned into *near-zone* and *wave-zone* domains. In order to do this, we can introduce the following characteristic time scale, $t_c$, the time required for significant changes to happen within the source function $\mu$. Typically, we expect $\dot{\mu} \sim \mu/t_c$ in the region covered by the source function. We can also define characteristic a frequency for the source and a characteristic wavelength for the radiation as

$$\omega_c = \frac{2\pi}{t_c} \ , \tag{3.24}$$

$$\lambda_c = \frac{2\pi}{\omega_c} \ . \tag{3.25}$$

The near zone is defined by the condition

$$r \ll \lambda_c \ , \tag{3.26}$$

the region for which $r = |\boldsymbol{x}|$ is much smaller than the characteristic wavelength $\lambda_c$, whilst the wave zone is defined by $r \gg \lambda_c$. In the near zone, the retarded time $t_r = t - r$ is very close to the coordinate time and *time derivatives are small* compared with spatial derivatives. By contrast, in the wave zone, the effects of the finite speed of field propagation are important since the retarded time differs significantly from the coordinate time, and spatial and time derivatives are of comparable size.

The *post-Newtonian expansion* is a method to approximate the near-zone contribution to the integral in Equation (3.23). For our application to cosmology, we will define $L_N$, the typical length scale associated with virialised large-scale structures in the real universe, and demand that

$$L_N \ll \lambda_c = \frac{2\pi}{\omega_c} = t_c \ , \tag{3.27}$$

where we take $\lambda_c$, the characteristic wavelength of null waves in cosmology, to be similar in scale to the particle horizon. The implication of this condition is that the typical velocities of sources in this regime are in some sense *slow*, since the characteristic dimensionless peculiar velocity will be given by

$$v_c \sim \frac{L_N}{t_c} \ll 1 \ . \tag{3.28}$$

We have therefore fulfilled the promise made earlier, where we stated that post-Newtonian expansions were slow-motion, weak-field approximations to general relativity. We can now consider the action of spatial and temporal derivatives on the source function:

$$\nabla\mu \sim \frac{\mu}{L_N} \,, \tag{3.29}$$
$$\dot{\mu} \sim \frac{\mu}{t_c} \tag{3.30}$$
$$\implies \dot{\mu} \ll |\nabla\mu|. \tag{3.31}$$

This result is consistent with the small velocities restriction; the implication is that the typical time variation of the sources is small compared to their spatial variation - exactly the physical conditions we expect in the virialised structures in the real universe. We should further note that the order of smallness of the time derivative is the same as the order of smallness of the characteristic peculiar velocity, $v_c$, i.e $\dot{\mu} \sim v_c|\nabla\mu|$. We can further consider the implications of this restriction on typical gravitational potentials, here represented by the symbolic quantity $\psi$. If $L_N \sim |\boldsymbol{x} - \boldsymbol{x}'| \ll t$, then a Taylor expansion of the time dependent part of Equation (3.23) yields the leading order term

$$\psi = \int_{\mathcal{V}} \frac{\mu(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3 x' \,, \tag{3.32}$$

where $\mathcal{V}$ is the spacelike 3-volume obtained by projecting $\mathcal{C}$ onto a hypersurface at constant $t$. By taking spatial and temporal derivatives of this leading order term, and using the results for $\dot{\mu}$ and $|\nabla\mu|$, we find that

$$\dot{\psi} \ll |\nabla\psi| \,. \tag{3.33}$$

Dimensional analysis of Equation (3.32) also reveals that $\psi \sim \mu L_N^2$.

Up to this point, these general considerations have followed from the assumption of small-scales and velocities, and the fact that the Einstein equation can be exactly reformulated as a null wave equation. We expect that the characteristic size of a generic gravitational potential should be $\psi \sim \mu L_N^2$. However, we are also supposed to be working within the confines of a weak-field expansion, which mandates that $\delta g_{\mu\nu}$, the metric perturbation, is in some sense small. At this point, we cannot continue using $\psi$ as a stand in for a generic gravitational potential, since there are many such potentials within $\delta g_{\mu\nu}$, and in general, although they must all be smaller than $O(\bar{g}_{\mu\nu})$, there is no requirement that all of these potentials have *the same order of smallness*. In particular, the field equations and geodesic equation link particular components of $\delta_{\mu\nu}$ to source terms that generate them. Therefore, one can directly deduce the characteristic size of different metric components by checking the sizes of the first terms that source them in the field and geodesic equations. In order to perform this exercise, it is useful to introduce the *post-Newtonian counting parameter*:

$$\eta \sim v \sim \frac{|\partial/\partial\tau|}{|\partial/\partial x|} , \tag{3.34}$$

where we have changed back to conformal time (which still has the same effect on order-of-smallness). We can use this smallness parameter to nominally keep track of the sizes of different quantities derived in this expansion. Let us first look at the geodesic equation for freely falling time-like particles, with 4-vector $u^\mu = (1, v^i)$ and a metric taking the form $g_{\mu\nu} = \bar{g}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\tau, \boldsymbol{x})$, where $\bar{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$ is the flat FLRW metric, and $\delta g_{\mu\nu}$ comprises many gravitational potentials with unknown sizes.

At leading order $u^\mu \nabla_\mu u^\nu = 0$ reduces to

$$\frac{\partial}{\partial\tau} v^i = \frac{1}{2} \bar{g}^{ij} \delta g_{00,j} . \tag{3.35}$$

The implication of this result is that the leading order perturbation to the metric has the characteristic size $\delta g_{00} \sim \eta^2$. If there were any position dependent terms that were larger than $\eta^2$ in this metric component, it would be inconsistent with a leading order equation of this form. On the other hand, the leading order component of the time-time Einstein field equation, given the stress energy tensor for a perfect fluid, takes the form

$$\nabla^2 \delta g_{00} \sim \rho , \tag{3.36}$$

where $\rho$ is the leading order energy density of matter fields. The implication is that the maximum size $\rho$ can have is $\rho \sim \eta^2 L_N^{-2}$. Furthermore, the obvious similarity between this relationship and the Newtonian Poisson equation motivates us to associate $\delta g^{(2)}_{00}$ (where the superscript $^{(2)}$ indicates that this quantity is of size $\sim \eta^2$) with the Newtonian gravitational potential, $2U$. This is further justified by noting that we have now recovered the typical virial-type relationship, $U \sim v^2$, expected in the gravitationally collapsed structures this expansion is supposed to model.

This type of expansion can be extended to higher orders, by expanding the stress energy tensor. In particular, we expand

$$\rho = \rho^{(2)} + \rho^{(4)} + \dots \ , \tag{3.37}$$

$$P = \bar{P}^{(2)} + \delta P^{(4)} + \dots \ , \tag{3.38}$$

resulting in an energy momentum tensor that takes the form (up to $\mathcal{O}(\eta^4 L_N^{-2})$)

$$T_{00}^{(2)} = -\bar{g}_{00}\rho^{(2)} \ , \tag{3.39}$$

$$T_{00}^{(4)} = -\bar{g}_{00}\rho^{(4)} + \rho^{(2)}\left(\bar{g}_{00}u^{(1)i}u_i^{(1)} + g_{00}^{(2)}\right) \ , \tag{3.40}$$

$$T_{0i}^{(3)} = -\sqrt{-\bar{g}_{00}}\rho^{(2)}u_i^{(1)} \ , \tag{3.41}$$

$$T_{ij}^{(4)} = (\rho^{(2)} + \bar{P}^{(2)})u_i^{(1)}u_j^{(1)} + \delta P^{(4)}g_{ij}^{(0)} \ . \tag{3.42}$$

It should be noted that there can be no spatially dependent pressure in a post-Newtonian expansion larger than $\sim \eta^4 L_N^{-2}$, but we are free to include a "background" homogeneous component at $\mathcal{O}(\eta^2 L_N^{-2})$, which we have done. In these expressions, the spatial part of the 4-velocity satisfies $u^{i(1)} \sim \eta$, and $\bar{g}_{0i} = 0$ in regular Euclidean comoving spatial coordinates for a flat FLRW geometry. Superscripts on dimensional quantities should be understood to carry the dimensional factors implicitly, e.g. $\rho^{(2)} \sim \eta^2 L_N^2$, whilst $g_{00}^{(2)} \sim \eta^2$. Given this stress energy tensor from which to construct source terms, consideration of the field and geodesic equations leads to a metric of the form

$$g_{00} = \bar{g}_{00}(\tau) + \delta g_{00}^{(2)}(\tau, \boldsymbol{x}) + \frac{1}{2}\delta g_{00}^{(4)}(\tau, \boldsymbol{x}) \dots \ , \tag{3.43}$$

$$g_{ij} = \bar{g}_{ij}(\tau) + \delta g_{ij}^{(2)}(\tau, \boldsymbol{x}) \ , \tag{3.44}$$

$$g_{0i}^{(3)} = \delta g_{0i}^{(3)} \ . \tag{3.45}$$

Since we identified $g_{00}^{(2)}$ with $U$, the Newtonian gravitational potential, the new quantities $g_{00}^{(4)}$, $g_{ij}^{(2)}$ and $g_{0i}^{(3)}$ are generally referred to as the "post-Newtonian potentials". Calculation of them constitutes specification of the metric to so-called first post-Newtonian, or "1PN" order. It should also be noted that the order required for application of the term "post-Newtonian" depends on which part of the metric is under consideration. This is because of the small-time derivative condition - spatial and time derivatives operate on different metric components in the geodesic equation.

The first spatially dependent term in $g_{0i}$ occurs at $\mathcal{O}(\eta^3)$. This is because the first source term for this potential is $\rho^{(2)}v^i \sim \eta^3 L_N^{-2}$. It should also be noted that certain terms that might be expected to appear in a generic higher-order expansion are absent, both in the metric and the stress energy tensor. For example, an obvious question is "Why is there no $\delta g_{00}^{(3)}$?". In principle, one could include this term; however, since there is no source term in the stress energy tensor for such a term, it would necessarily satisfy a homogeneous Poisson type equation, with the same differential operator as the left hand side of the equation for $\delta g_{00}^{(2)}$. This means that the $\delta g_{00}^{(3)}$ term can be subsumed into the definition $g_{00}^{(2)}$ without loss of generality, and that, without sources, a term like $\delta g_{00}^{(3)}$ describes no new physics. It is therefore excluded for simplicity.

These considerations have lead us to specific forms for a metric and stress energy tensor in Equations (3.43) and (3.39), together with an additional rule that time derivatives of quantities add additional factors of the post-Newtonian smallness factor, $\eta$. Whilst we could in principle apply the iterative procedure detailed in [51], for our purposes it will be sufficient to simply look at the field equations directly, selecting terms that have the same order of smallness. The smallness of time derivatives compared to spatial derivatives implies that the field equations that would normally correspond to null wave equations can instead be written at leading-order as Poisson equations:

$$\Box \delta g_{\mu\nu} \propto T_{\mu\nu} \quad \Rightarrow \quad \nabla^2 \delta g_{\mu\nu} \propto T_{\mu\nu}\,, \tag{3.46}$$

where $\Box = \bar{g}^{\mu\nu}\partial_\mu \partial_\nu$ and $\nabla^2 = \bar{g}^{ij}\partial_i \partial_j$.

The support for the integral that gives the function $\delta g_{\mu\nu}(\tau, \boldsymbol{x})$ in Eq. (3.46) should really be taken to be on the past light cone $\mathcal{L}$ of the point $P$ at position $\boldsymbol{x}$. This shows the causal nature of general relativity - gravitational interactions propagate at the speed of light. However, such an approach would be problematic to apply in cosmology, as the integral for the gravitational fields at each point in space would have its own distinct domain (i.e. its own past lightcone). A fortunate consequence of the slow-motion expansion is that on scales $r \lesssim r_{\rm c}$, where $r_c$ is a characteristic distance scale satisfying $r_c/\lambda_c \sim 10^{-2}$,we can approximate the past light cone of a point as being given by a space-like surface $\mathcal{S}$ of constant $\tau$ [51], as shown in Fig. 3.1. This is because the conformal time taken for a null signal to go from one side of such a domain to the other is negligible compared to $\tau_{\rm c}$ (the characteristic conformal time), and means that we can find solutions for $\delta g_{\mu\nu}(\tau, \boldsymbol{x})$ at some time $\tau$ by simply integrating over a suitable region of a hypersurface of constant $\tau$. The integrals for the gravitational field value at neighbouring points in space then have their support on overlapping domains, and the whole process of finding solutions is considerably simplified.

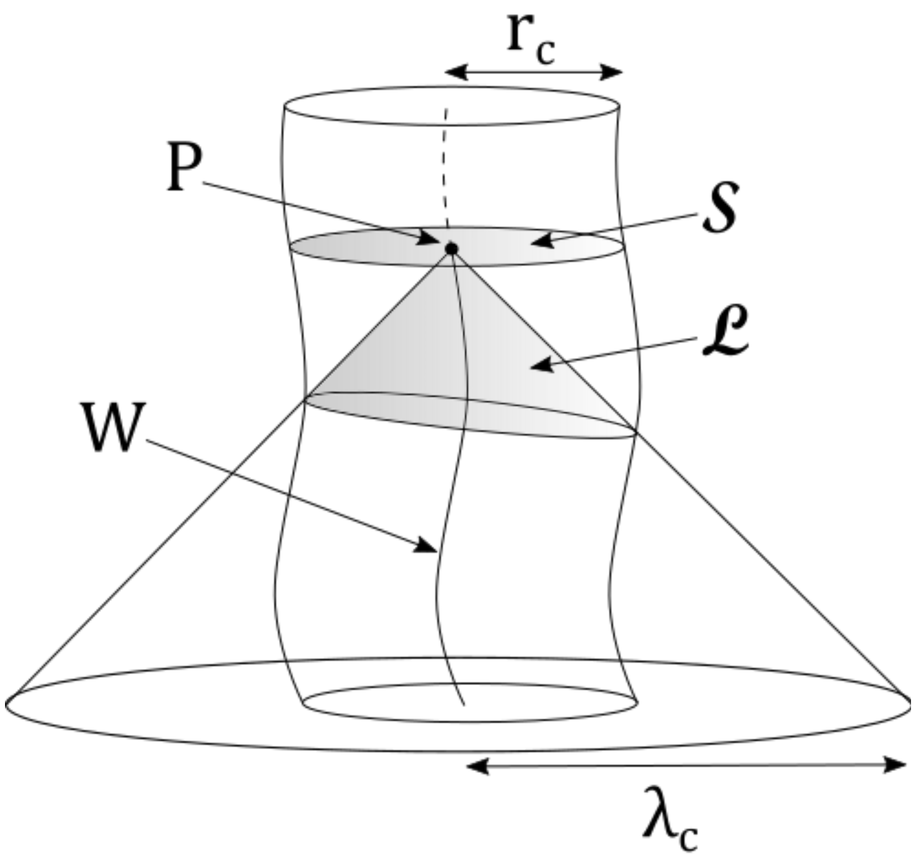


Figure 3.1.: The past lightcone $\mathcal{L}$ of a point $P$ following a worldline $W$. The support for the metric perturbations at $P$ can be approximated as being located on the space-like hypersurface $\mathcal{S}$, as long as $r_{\mathrm{c}} \ll \lambda_{\mathrm{c}}$.

At this point, one might think we are ready to write down field equations, however, we must first address the so-called "gauge problem" in post-Newtonian gravity.

### 3.2.4. Post-Newtonian gauge problem

Our background metric is $\bar{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$, and we will Helmholtz decompose our perturbations into their irreducible representations as

$$\delta g_{00} = -2a^2\phi \tag{3.47}$$

$$\delta g_{0i} = a^2(B_{,i} - S_i) \tag{3.48}$$

$$\delta g_{ij} = a^2(-2\psi\,\delta_{ij} + 2E_{,ij} + 2F_{(i,j)} + h_{ij})\,, \tag{3.49}$$

where $S_i$, and $F_i$ are divergence-free vector fields of magnitude $\sim \eta^3$, $h_{ij}$ is a trace-free and divergence-free tensor field of magnitude $\sim \eta^4$, and $\psi$, $\phi$, $E$ and $B$ are scalar fields of magnitude $\sim \eta^2$. Degrees of freedom should be taken to have the appropriate dimensions in $L_N$ such that they are dimensionless when they appear in the metric, e.g $E \sim \eta^2 L_N^2$ such that $\partial_i \partial_j E \sim \eta^2$.

The coordinates used to express the background FLRW metric are related to the existence of a preferred observer frame, one that is at rest with respect to the CMB, and are in some sense unique, at least up to spatial rotations and translations. Unfortunately, the coordinates we use to describe $\delta g_{\mu\nu}$ are not unique - residual coordinate freedom allows one to make infinitesimal coordinate transformations of the form

$$x^\mu \to x^\mu + \xi^\mu \, , \tag{3.50}$$

leaving physical results unchanged.

There is another, more geometrical way to understand this state of affairs. Up until this point, we have glossed over the fact that $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ actually live on two completely different manifolds. This raises conceptual questions regarding the nature of objects like

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu} \, , \tag{3.51}$$

which clearly involves two objects defined on different spacetime manifolds. It is not possible to add and subtract tensor defined on different manifolds, without first specifying a *map* that associates objects defined on one manifold to objects defined on the other. This map allows us to make sense of expressions like Equation (3.51), by first using the map to "transport" $g_{\mu\nu}$ to the background spacetime manifold (or vice versa). The irreducibly decomposed fields comprising $\delta g_{\mu\nu}$ can then be thought of as scalar, vector and tensor fields "living" on the background space, much in the same way as the fields in typical classical field theories exist independently of the spacetime on which they live. In this way, weak field GR expansions reduce the problem of solving the evolution of a dynamic spacetime to that of solving a classical gauge theory on some predetermined background metric (which is an exact solution of the field equations), and understanding the "fields" in this gauge theory to describe small perturbations to the spacetime itself.

The residual coordinate freedom described above can also be conceptualised as the manifestation of the *non-uniqueness* of the map between unperturbed and perturbed spacetimes, $(\bar{\mathcal{M}}, \bar{g}_{\mu\nu})$ and $(\mathcal{M}, g_{\mu\nu})$. One is therefore forced to make a choice of coordinates in which to express the fluctuations $\delta g_{\mu\nu}$ - this choice is referred to as a *gauge choice* and is necessary in order to reduce the number of degrees of freedom and close the system. Choosing a gauge also prevents the occurrence of spurious gauge modes in the solutions. This gauge symmetry of the “fields” in our classical field theory is inherited from the full diffeomorphism covariance of nonperturbative general relativity.

A gauge transformation can be said to be either *active* or *passive*. We will use the former of these approaches, which changes the point in space-time that a given set of coordinate values identifies. The action of such a transformation on a tensor field $\boldsymbol{T}$ can be written in the form

$$\boldsymbol{T} \rightarrow \tilde{\boldsymbol{T}} = e^{\mathcal{L}_{\boldsymbol{\xi}}} \boldsymbol{T}\,, \tag{3.52}$$

where $\mathcal{L}_{\boldsymbol{\xi}}$ is the Lie derivative with respect to the gauge generator $\boldsymbol{\xi}$, and where a tilde denotes the field $\boldsymbol{T}$ after the transformation. The exponential map is used here to ensure that the group structure of the diffeomorphisms associated with the transformations is preserved.

Treating the coordinates on the manifold as a set of four scalar fields, the gauge transformation in Eq. (3.52) can be seen to be equivalent to

$$x^\mu(p) \rightarrow x^\mu(q) = e^{\xi^\alpha \partial_\alpha|_p} x^\mu(p)\,, \tag{3.53}$$

where $x^\mu = x^\mu(p)$ on the right-hand side is evaluated at some point $p$, while $x^\mu(q)$ is evaluated at the point $q$ located along the flow of the gauge generator field $\boldsymbol{\xi}$ from $p$. This construction is exactly the map we are searching for, identifing a point in the background space-time with a point in the perturbed space-time.

If we apply the transformation in Eq. (3.52) to the metric we obtain:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \mathcal{L}_\xi\, g_{\mu\nu} + \frac{1}{2}\mathcal{L}_\xi^2\, g_{\mu\nu} + \,\dots\,, \tag{3.54}$$

which gives the various components of the metric transforming as

$$g_{00} \rightarrow g_{00} + \xi^\mu \partial_\mu g_{00} + 2 g_{0\mu}\, \dot{\xi}^\mu + \dots \tag{3.55}$$

$$g_{0i} \rightarrow g_{0i} + \xi^\mu \partial_\mu g_{0i} + g_{0\mu}\, \xi^\mu_{,i} + g_{i\mu}\, \dot{\xi}^\mu + \dots \tag{3.56}$$

$$g_{ij} \rightarrow g_{ij} + \xi^\mu \partial_\mu g_{ij} + 2 g_{\mu(i}\, \xi^\mu{}_{,j)} + \dots \tag{3.57}$$

Let us now consider linear gauge transformations of the metric, as given to linear order in $\xi^\mu$ by Eqs. (3.54)-(3.57). First, consider the size of each of the terms that results from the gauge transformation in Eq. (3.54). Starting with the $ij$-component of the metric, we can see that the transformation (3.57) has terms of magnitude

$$g_{ij} \rightarrow g_{ij} + O(\xi^i) + O(\eta\, \xi^0)\,, \tag{3.58}$$

where we use $O(x)$ to mean terms of order $x$ or smaller in the post-Newtonian expansion. In deriving this expression we have used the rules for the order-of-magnitude of each of the components of the metric, and the relative size of their derivatives, as outlined in Section 3.2.3. We have also taken the components of the gauge generator $\xi^\mu$ to obey the same rules with respect to derivative operators (i.e. that time derivatives of these objects are small compared to space derivatives).

Performing the same analysis for the $0i$-component of the metric we find

$$g_{0i} \rightarrow g_{0i} + O(\eta\, \xi^i) + O(\xi^0)\,. \tag{3.59}$$

In order for the gauge transformed $ij$ and $0i$-components of the metric to be no larger than $\eta^2$ and $\eta^3$, respectively, we can see that we must have

$$\xi^i \sim \eta^2 \qquad \text{and} \qquad \xi^0 \sim \eta^3\,. \tag{3.60}$$

If the former of these conditions was violated, and the magnitude of $\xi^i$ were allowed to be larger than $\eta^2$, then it can be seen that the gauge transformed $ij$-components of the metric would have terms larger than $\eta^2$. This would mean that they would be larger than allowed in the post-Newtonian expansion of the metric, and the transformation would not be part of the gauge group of the theory. Similarly, if the magnitude of $\xi^0$ were allowed to be any larger than $\eta^3$ then the gauge transformed $0i$-components of the metric would contain parts that were larger than $\eta^3$, which is also forbidden for the same reason.

We have already seen that the perturbations to different components of the metric can have leading-order parts with different orders of magnitude. If we investigate the leading-order terms that are generated in the transformation of the 00-component of the metric we now find

$$g_{00} \rightarrow g_{00} + O(\eta^2\, \xi^i) + O(\eta\, \xi^0)\,, \tag{3.61}$$

which, using Eq. (3.60), can be seen to be equivalent to

$$g_{00} \to g_{00} + O(\eta^4)\,. \tag{3.62}$$

This means that the leading-order perturbation to the 00-component of the metric, which exists at order $\eta^2$, is entirely unchanged by the gauge transformations that this theory admits, and that only sub-leading terms are affected. This result severely limits what can be done with gauge transformations when using post-Newtonian expansions.

Having identified the orders of magnitude of the leading-order parts of the gauge generators, we can now find the leading-order parts of the gauge transformations of each of the degrees of freedom in the metric. These are given by

$$g_{00} \to g_{00} + \xi^0 \dot{g}_{00} + \xi^i g_{00,i} + 2 g_{00}\, \xi^{0\prime} + O(\eta^5) \tag{3.63}$$

$$g_{0i} \to g_{0i} + g_{00}\, \xi^0_{,i} + g_{ij}\, \xi^{j\prime} + O(\eta^4) \tag{3.64}$$

$$g_{ij} \to g_{ij} + 2 g_{k(i}\, \xi^k{}_{,j)} + O(\eta^3)\,, \tag{3.65}$$

which gives

$$\phi \to \phi + \mathcal{H}\, \xi^0 + \xi^{0\prime} + \phi_{,i}\xi^i \tag{3.66}$$

$$B \to B + \zeta' - \xi^0 \tag{3.67}$$

$$S_i \to S_i - \zeta_i' \tag{3.68}$$

$$\psi \to \psi \tag{3.69}$$

$$E \to E + \zeta \tag{3.70}$$

$$F_i \to F_i + \zeta_i \tag{3.71}$$

$$h_{ij} \to h_{ij}\,, \tag{3.72}$$

where we have kept terms up to order $\eta^4$ in $\phi$, as this is the order required to obtain the post-Newtonian equations of motion for massive test particles, and where we have decomposed the vector gauge generator $\xi_i = \zeta_i + \zeta_{,i}$ into a divergence-free vector piece ($\zeta_i$) and the gradient of a scalar ($\zeta_{,i}$). The reader will note that as well as the leading-order part of $\phi$ (at order $\eta^2$) being gauge invariant, the same can also be said of the leading-order parts of $\psi$ and $h_{ij}$.

In terms of the variables defined in (3.47) we can expand the matter 4-velocity vector as

$$u^\mu = \frac{1}{a}\left(1 - \phi + \frac{1}{2}v^2, v^i\right) , \tag{3.73}$$

where $v^i$ is the matter 3-velocity and $v^2 = v^i v_i$. Expanding the stress-energy tensor in the parameter $\eta$ we find

$$T^0{}_0 = -\rho(1+v^2) + O(\eta^5) \tag{3.74}$$

$$T^0{}_i = \rho v_i + O(\eta^4) \tag{3.75}$$

$$T^i{}_j = \delta^i{}_j \delta P + (\rho + \bar{P}) v^i v_j + O(\eta^5) , \tag{3.76}$$

which under the gauge transformation (3.52) gives

$$\mu \to \mu \tag{3.77}$$

$$\Pi \to \Pi + \xi^i (\ln \mu)_{,i} \tag{3.78}$$

$$P \to P \tag{3.79}$$

$$v^i \to v^i , \tag{3.80}$$

where we have written $\rho = \mu(1+\Pi)$, such that $\mu \sim \eta^2$ is the rest-mass density and $\Pi \sim \eta^2$ is the specific energy density. All lowest-order parts of the matter variables can be seen to transform trivially, with an additional term at order $\eta^4$ appearing in the transformation of $\Pi$. $P$ can be split into a "background" homogeneous piece $\bar{P}$, which can be of maximum size $\eta^2 L_N^{-2}$, and an inhomogeneous piece, $\delta P \sim \eta^4 L_N^{-2}$, which can be no larger than this size in a post-Newtonian expansion.

Now we have assembled all the gauge transformations of the various different quantities in the theory and are free to select the components $\xi^\mu$ as we please. Usually, one selects the gauge generators either such that certain degrees of freedom in the metric or matter variables vanish, or such that some potentially desirable physical property is realised. We will discuss the viability of various gauges in cosmologies with nonlinear structure in conjunction with consideration of the analogous problem in cosmological perturbation theory. For now though, we will restrict ourselves to *longitudinal gauge*, defined by the choice

$$B = E = F_i = 0 , \tag{3.81}$$

which can be realised by choosing

$$\xi^0 = B + E' \,, \tag{3.82}$$

$$\zeta = -E \,, \tag{3.83}$$

$$\zeta_i = -F_i \,. \tag{3.84}$$

This gauge diagonalises the scalar part of the metric, which considerably simplifies many calculations. It is sometimes also referred to as the "zero-shear gauge", since the shear, defined by $\sigma = E' - B$ identically vanishes in this gauge.

## 3.3. Field equations and equations of motion

### 3.3.1. Leading order

If we write the line-element in as a weak-field perturbation of FLRW, as in Eqs. (3.47), then the leading-order part of the $ij$ field equation can be written as

$$\begin{aligned} a^2\, R^{(2)i}{}_j &= \nabla^2\psi\, \delta^i{}_j - (\phi-\psi)_{,ij} + \left(2\mathcal{H}^2 + \mathcal{H}'\right)\delta^i{}_j - \nabla^2 h_{ij} \\ &= 4\pi\,\mu\, a^2\, \delta^i{}_j + \Lambda\, a^2\, \delta^i{}_j\,, \end{aligned} \tag{3.85}$$

where the superscript in $R^{(2)i}{}_j$ indicates that this is the part of this tensor at order $\eta^2$ in the $v/c$ expansion, in appropriately chosen units. This equation immediately tells us that

$$\partial_i\partial^j(\psi-\phi) - \frac{1}{3}\delta^j{}_i = 0 \qquad \Rightarrow \qquad \boxed{\psi^{(2)} = \phi^{(2)}}\,, \tag{3.86}$$

and

$$\nabla^2 h^{(2)}_{ij} = 0 \qquad \Rightarrow \qquad \boxed{h^{(2)}_{ij} = 0}\,, \tag{3.87}$$

where appropriate boundary conditions have been used to infer the results on the right.

The leading order longitudinal gauge scalar field equations (including the cosmological constant) are then:

$$\mathcal{H}^2 + \frac{2}{3}\nabla^2\psi = \frac{8\pi}{3}\,\mu\, a^2 \, + \frac{\Lambda}{3}\, a^2 + O(\eta^4\, L_N^{-2}) \tag{3.88}$$

and

$$\mathcal{H}' - \frac{1}{3}\nabla^2\phi = -\frac{4\pi}{3}(\mu + 3\bar{P})\, a^2 + \frac{\Lambda}{3}\, a^2 + O(\eta^4\, L_N^{-2}) \,, \tag{3.89}$$

where $\mathcal{H} \sim \tau_c^{-1} \sim \eta$ and $\mathcal{H}' \sim \tau_c^{-2} \sim \eta^2$ (in units such that $r_c \sim 1$), and where dashes indicate differentiation with respect to conformal time (which still adds factors of $\eta$ since $d\tau = a\, dt$) . These equations are a combination of the Hubble equations and the Newton-Poisson equations for $\phi$ and $\psi$, which both occur at the same order in this expansion. Within a region of space $\mathcal{S}$, of scale $r \lesssim r_c$, they can be transformed to the usual Newtonian equations through a suitable choice of coordinates. It is also known that many such regions can be patched together to form a cosmology described by a line-element that is close to a single global FLRW solution [53].

If we integrate Eqs. (3.88)-(3.89) over $\mathcal{S}$, and divide by the spatial volume of that region, we recover the standard Friedmann equations, as well as the Newton-Poisson equation on an expanding background,

$$\nabla^2\phi = \nabla^2\psi = \nabla^2 U = 4\pi\, \delta\mu\, a^2 \,, \tag{3.90}$$

as long as we choose the boundary condition

$$\int_{\partial\mathcal{S}} \boldsymbol{\nabla} U \cdot d\boldsymbol{S} = 4\pi G \,\langle\delta\mu\rangle\, a^2 \,, \tag{3.91}$$

where we have written $\mu = \bar{\mu} + \delta\mu$, where $\langle\delta\mu\rangle$ is the volume averaged value of $\delta\mu$ in the region $\mathcal{S}$, and where we now replace $\psi$ and $\phi$ with the standard Newtonian gravitational potential, $U$, where $U$ satisfies $\nabla^2 U = 4\pi\delta\mu\, a^2$, i.e.

$$U(\boldsymbol{x}, \tau) = -a^2(\tau) \int \frac{\delta\mu(\tilde{\boldsymbol{x}}, \tau)}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|}\, d^3\tilde{x} \,, \tag{3.92}$$

where appropriate boundary conditions have again been applied.

It is important to note that there is no assumption made about the relative sizes of $\bar{\mu}$ and $\delta\mu$ here; the post-Newtonian expansion is specifically constructed to allow for large density contrasts to be consistently modelled, and this means that $\delta\mu/\bar{\mu}$ is allowed to be much larger than one without signalling any breakdown in the weak-field expansion. It is also important to note that if this boundary condition cannot be realised, the implication is that the expansion is incompatible with the choice of an FLRW background metric. This can be seen by considering the spatial dependencies of terms of in Equations (3.88) and (3.89) - if the Poisson equation is not satisfied, these equations would imply spatially dependent Hubble rate, which is inconsistent with the original assumptions.

The left-hand side of Eq. (3.91) can be set to zero if one chooses $\mathcal{S}$ to have periodic boundary conditions, which also sets the right-hand side to zero (as the average of this spatial domain would automatically be equal to the global average of the cosmology). In general, it seems conceivable that Eq. (3.91) may not be satisfied. If this is so, then one should expect strong cosmological back-reaction, and a violation of our initial ansatz of a perturbed FLRW space-time, but we will not consider this further here.

It is noteworthy that the Friedmann equations and the Newton-Poisson equations occur at the same order of magnitude in this expansion. This shows the well known fact that post-Newtonian expansions are *not* (strictly speaking) a direct application of perturbation theory, a fact that is already obvious from the leading-order conservation equations:

$$\rho' + 3\mathcal{H}\rho + \partial_i \left(\rho v^i\right) = 0 \,, \tag{3.93}$$

and

$$\rho v_j' + \rho v^i \partial_i v_j + \rho \mathcal{H} v_j = -\rho \partial_j U - \partial_j \delta P \,, \tag{3.94}$$

which are the standard equations of Newtonian gravity on an expanding background. These equations are clearly nonlinear, and therefore cannot be considered as being the result of an application of perturbation theory (where small nonlinear effects are modelled by a hierarchy of *linear* differential equations with inhomoegenous source terms). Nevertheless, they are well-defined, and the post-Newtonian expansion itself constitutes a well-defined expansion of the field and conservation equations, which has been extensively applied in other areas of gravitational physics.

All equations in this section, as well as higher-order equations, can be obtained by direct coordinate transformation from their form in the post-Minkowski approach [53]. Their existence shows the direct correspondence (through an isomorphism) of the expansion about a Friedmann space and the extremely well studied expansions that are usually performed around Minkowski space. This isomorphism can be used to further justify the order of magnitude we have associated with the various quantities we have considered, as well as to understand some of the features of this approach to weak-field gravity that have now become apparent. Firstly, the applicability to scales $r \ll \mathcal{H}^{-1}$ can be seen to correspond directly to the requirement that $v \ll c$. Secondly, the mixing of Friedmann and Poisson equations can be shown to be a result of the leading-order part of the cosmological expansion arising from the motion of particles under the influence of Newtonian gravitational fields in the perturbed Minkowski approach. We refer the reader to Refs. [53, 83] for further details of these results.

### 3.3.2. Solving post-Newtonian equations in arbitrary gauge

If we consider the leading-order part of the $0j$ field equation, we find

$$\begin{aligned} -a^2\, {R^{(3)0}}_j &= \frac{1}{2}\nabla^2 S_i + \frac{1}{2}\nabla^2 F'_j + 2U'_{,j} + 2\mathcal{H}U_{,j} \\ &= -8\pi\mu\, v_j\, a^2\,, \end{aligned} \tag{3.95}$$

where we have used the results in Eq. (3.90). Solving this equation we find

$$\boxed{S_j^{(3)} + F_j'^{(2)} = -2(V_j + W_j)}\,, \tag{3.96}$$

where the potentials on the right-hand side are given by

$$V_j = -a^2(\tau) \int \frac{\mu(\tilde{\boldsymbol{x}},\tau)\, v_j(\tilde{x})}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|}\, d^3\tilde{x}\,, \tag{3.97}$$

and

$$W_j = -a^2(\tau) \int \frac{\mu(\tilde{\boldsymbol{x}},\tau)\, \boldsymbol{v}(\tilde{x}) \cdot (\boldsymbol{x} - \tilde{\boldsymbol{x}})(x - \tilde{x})_j}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|^3}\, d^3\tilde{x}\,, \tag{3.98}$$

and where we have used the result $U'_{,j} + \mathcal{H}U_{,j} = \frac{1}{2}\nabla^2(W_j - V_j)$, which can be proven using the continuity equation.

Let us now consider the 00 field equation in the case where $P = 0$. To order $\eta^2$, and using the results above, this equation gives

$$-3\mathcal{H}' = 4\pi\,\bar{\mu}\, a^2 - \Lambda\, a^2\,, \tag{3.99}$$

which can clearly be seen to correspond to the second Friedmann equation in (2.70), and which together with the relation

$$\mathcal{H}' + 2\mathcal{H}^2 = 4\pi a\bar{\mu} + \Lambda a^2\,, \tag{3.100}$$

derived by taking the trace of Eq. (7.3), gives the first Friedmann equation (2.69).

The same component of the Ricci tensor to order $\eta^4$ gives

$$
\begin{aligned}
-a^2 \, R^{(4)0}{}_0 =& \nabla^2\phi^{(4)} + \nabla^2 B' + \mathcal{H}\nabla^2 B + 3U'' \\
& - U_{,j}\nabla^2 F_j - 2F_{(j,k)}U_{,jk} - 2U_{,j}U_{,j} \\
& - \nabla^2 E'' - \mathcal{H}\nabla^2 E' - U_{,j}\nabla^2 E_j \\
& - 2U_{,jk}E^{(2)}_{,jk} + 6\mathcal{H}U' + 6\mathcal{H}'U \,.
\end{aligned}
$$

This result can now be used with the relevant field equation,

$$
-a^2 \, R^{(4)0}{}_0 = 4\pi \, \mu \, a^2 \left( 2v^2 + \Pi + 3\frac{P}{\mu} \right) \,, \tag{3.101}
$$

to find

$$
\boxed{
\begin{aligned}
& \phi^{(4)} + B^{(3)\prime} + \mathcal{H}B^{(3)} - U^2 - U_{,j}F^{(2)}_j \\
& \qquad\qquad - E^{(2)\prime\prime} - \mathcal{H}E^{(2)\prime} - U_{,j}E^{(2)}_{,j} \\
= \; & \frac{1}{2}\Phi_1 + 3\Phi_2 - 5\delta\Phi_2 + \Phi_3 + 3\Phi_4 \\
& \qquad\qquad - \delta\Phi_{5j,j} - \delta\Phi_6 + \frac{3}{2}\mathcal{A} + \frac{3}{2}\mathcal{B} \,,
\end{aligned}
}
$$

where

$$
\begin{aligned}
\mathcal{A} &= -a^2(\tau) \int \frac{\tilde{\mu}[\tilde{\boldsymbol{v}} \cdot (\boldsymbol{x} - \tilde{\boldsymbol{x}})]^2}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|^3} \, d^3\tilde{x} \\
\mathcal{B} &= -a^2(\tau) \int \frac{\tilde{\mu}}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|} (\boldsymbol{x} - \tilde{\boldsymbol{x}}) \cdot \frac{d\tilde{\boldsymbol{v}}}{d\tau} \, d^3\tilde{x} \\
\Phi_1 &= -a^2(\tau) \int \frac{\tilde{\mu} \, \tilde{v}^2}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|} \, d^3\tilde{x} \\
\Phi_2 &= -a^2(\tau) \int \frac{\tilde{\mu} \, \tilde{U}}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|} \, d^3\tilde{x} \\
\delta\Phi_2 &= -a^2(\tau) \int \frac{\delta\tilde{\mu} \, \tilde{U}}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|} \, d^3\tilde{x} \\
\Phi_3 &= -a^2(\tau) \int \frac{\tilde{\mu} \, \tilde{\Pi}}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|} \, d^3\tilde{x} \\
\Phi_4 &= -a^2(\tau) \int \frac{\tilde{P}}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|} \, d^3\tilde{x} \\
\delta\Phi_{5j} &= -a^2(\tau) \int \frac{\delta\tilde{\mu} \, \tilde{F}_j}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|} \, d^3\tilde{x} \\
\delta\Phi_6 &= -a^2(\tau) \int \frac{\delta\tilde{\mu}_{,j} \tilde{E}_{,j}}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|} \, d^3\tilde{x} \,,
\end{aligned}
$$

where the variables adorned with the " ˜ " symbol are understood to be functions of the $\tilde{x}$ spatial coordinates, e.g. $\tilde{U} = U(\tilde{x}, \tau)$. These are all standard potentials used in post-Newtonian gravity, with the exceptions of $\delta\Phi_2$, $\delta\Phi_{5j}$ and $\delta\Phi_6$, which we have introduced here.

In deriving this last result we have used the following identities:

$$\begin{aligned}
2U_{,j}U_{,j} - 8\pi\bar{\rho}\, U\, a^2 &= \nabla^2 \left(U^2 - 2\Phi_2\right) \\
-U_{,j}\nabla^2 F_j - 2F_{(j,k)}U_{,jk} &= \nabla^2 \left(\delta\Phi_{5j,j} - U_{,j}F_j\right) \\
-U_{,j}\nabla^2 E_j - 2U_{,jk}E_{,jk} &= -\nabla^2 \left(U_{,j}E_{,j} - \delta\Phi_6\right) \\
U'' + 2\mathcal{H}U' + \left(\mathcal{H}^2 + \mathcal{H}'\right) U &= -\frac{1}{2}\nabla^2 \left(\mathcal{A} + \mathcal{B} - \Phi_1\right) ,
\end{aligned}$$

the last of which is proven using the continuity equation. We will also use the following identities in Section 5.3 of this thesis:

$$U' + \mathcal{H}U = -V_{j,j} = W_{j,j} \tag{3.102}$$

$$V_{[j,k]} = W_{[j,k]} \tag{3.103}$$

$$V_j' - W_j' = -\mathcal{H}\left(V_j - W_j\right) + \mathcal{A}_{,j} + \mathcal{B}_{,j} - \Phi_{1,j} \, . \tag{3.104}$$

All identities can be proven under the assumption that boundary terms vanish, as would occur (for example) in a space with periodic boundary conditions.

### 3.3.3. Post-Newtonian equations of motion

The equations of motion of post-Newtonian gravity can be obtained by expanding the conservation equations:

$$T^{\mu\nu}{}_{;\nu} = 0 \, , \tag{3.105}$$

which can be conveniently written as

$$\partial_\nu \left(\sqrt{-g}\, T^{\mu\nu}\right) + \Gamma^\mu{}_{\rho\nu}\sqrt{-g}\, T^{\rho\nu} = 0 \, , \tag{3.106}$$

where $g$ is the determinant of the metric. The metric in Eqs. (3.47), together with the longitudinal gauge condition gives the components of the stress-energy tensor to the required order as

$$
\begin{aligned}
T^{00} &= \frac{1}{a^2}\mu\left(1+v^2-2U+\Pi\right)+O(\eta^5)\\
T^{0i} &= \frac{1}{a^2}\mu\, v^i\left(1+\frac{1}{2}v^2-U+\Pi\right)+\frac{1}{a^2}P\, v^i+O(\eta^6)\\
T^{ij} &= \frac{1}{a^2}\left(\mu v^i v^j+P\delta^{ij}\right)\\
&\quad+\frac{1}{a^2}\left[(\mu\Pi+p)v^i v^j+2Up\delta^{ij}-2(E_{,}{}^{i}+F^i)P\right]+O(\eta^7)\,. \qquad (3.107)
\end{aligned}
$$

Likewise, the connection coefficients, up to the required order, are given by

$$
\begin{aligned}
\Gamma^0{}_{00} &= \mathcal{H}+U'+O(\eta^4) && (3.108)\\
\Gamma^0{}_{0i} &= U_{,i}+O(\eta^3) && (3.109)\\
\Gamma^0{}_{ij} &= \delta_{ij}\mathcal{H}+O(\eta^2) && (3.110)\\
\Gamma^j{}_{00} &= U_{,j}+\phi^{(4)}_{,j}+B'_{,j}+\mathcal{H}B_{,j}-S'_j-\mathcal{H}S_j && (3.111)\\
&\quad+2UU_{,j}-2E_{,ij}U_{,i}-2F_{(i,j)}U_{,i}+O(\eta^5) && (3.112)\\
\Gamma^j{}_{0k} &= \delta_{jk}(\mathcal{H}-U')-S_{[j,k]}+E'_{,jk}+F'_{(j,k)}+O(\eta^4) && (3.113)\\
\Gamma^j{}_{kn} &= -\delta_{jn}U_{,k}-\delta_{jk}U_{,n}+\delta_{kn}U_{,j}\\
&\quad+E_{,jkn}+F^j{}_{,nk}+O(\eta^3)\,, && (3.114)
\end{aligned}
$$

and the square root of the determinant of the metric is

$$
\sqrt{-g}=a^4(1-2U+\nabla^2 E)+O(\eta^3)\,.
$$

In deriving all of these equations we have used the results from Eqs. (3.87) and (3.90) to eliminate $h^{(2)}_{ij}$, and to write $\phi^{(2)}$ and $\psi^{(2)}$ in terms of $U$.

The order $\eta^3$ part of the time component of Eq. (3.106) can immediately be seen to reproduce the Newtonian equation of mass conservation on an expanding background, as given in Eq. (3.93). Likewise, the order $\eta^4$ of the spatial components of Eq. (3.106) gives the momentum conservation equation from (3.94), once we set $\phi=U$. The next non-vanishing contributions to the conservation equations (3.106) come at order $\eta^5$ in the time component, and order $\eta^6$ in the space components. These correspond to first post-Newtonian order, in the normal language of this type of weak-field expansion (though on a cosmological background here). We will now consider each of these in turn.

Calculating the order $\eta^5$ part of the time component of Eq. (3.106), and simplifying using the momentum conservation equation (3.94), gives

$$\begin{aligned}0 = \partial_\tau&\left[a^3\mu\left(\frac{1}{2}v^2 - 3U + \Pi + \nabla^2 E\right)\right]\\&+ \partial_j\left[a^3\mu\, v^j\left(-2U + \Pi + \nabla^2 E\right)\right]\\&+ a^3 p\left(v^j{}_{,j} + 3\mathcal{H}\right) .\end{aligned} \tag{3.115}$$

Calculating the order $\eta^6$ part of the spatial components of Eq. (3.106), and taking $p = 0$, gives

$$\begin{aligned}0 = \partial_\tau&\left[a^4\mu v^j\left(\frac{1}{2}v^2 - 3U + \Pi + \nabla^2 E\right)\right] + \partial_k\left[a^4\mu v^j v^k\left(-2U + \Pi + \nabla^2 E\right)\right]\\&+ a^4\mu\Big[U_{,j}\left(2v^2 - 4U + \Pi + \nabla^2 E\right) + \phi^{(4)}_{,j} + \mathcal{H}B_{,j} + B'_{,j}\\&\qquad - \mathcal{H}S_j - S'_j + 2UU_{,j} - 2E_{,ij}U_{,i} - 2F_{(i,j)}U_{,i}\Big]\\&- 2a^4\mu v^k\left[S_{[j,k]} + \delta_{jk}U' - E'_{,jk} - F'_{(j,k)} + v^j U_{,k} - \frac{1}{2}v^n E_{,jkn} - \frac{1}{2}v^n F^j{}_{,kn}\right] .\end{aligned} \tag{3.116}$$

These are all of the equations that are required to calculate the trajectories of test particles to first post-Newtonian order.

# 4. Perturbation Theory

In this chapter, we will introduce the various methods used to approximate solutions to the Einstein Field Equations that model mildly inhomogeneous, anisotropic universes. We will consider two related approximation schemes: Newtonian perturbation theory, relativistic perturbation theory. Mathematical techniques introduced at this stage will find subsequent use when we come to consider solutions to the two-parameter perturbation theory that is developed later.

## 4.1. Newtonian perturbation theory

The equations

$$\rho' + 3\mathcal{H}\rho + \partial_i\left(\rho v^i\right) = 0\,, \tag{4.1}$$

$$\rho v_j' + \rho v^i \partial_i v_j + \rho\mathcal{H} v_j = -\rho\partial_j U - \partial_j \delta P\,, \tag{4.2}$$

$$\nabla^2 U = 4\pi a^2 \delta\rho\,, \tag{4.3}$$

were derived in the previous chapter as the leading order inhomogeneous gravitational field equation, and the leading order stress-energy conservation equations in a post-Newtonian expansion around the FLRW metric, given a perfect fluid stress energy tensor. They also, unsurprisingly given the nomenclature, are the exact equations generally considered in *Newtonian cosmology* - an approach to cosmology that substitutes full general relativity with Newtonian gravity. Post-Newtonian gravity then enables one to compute successive relativistic corrections to exact Newtonian gravity. It should be noted that the Newtonian problem is still nonlinear because of the nonlinear terms in the Euler and continuity equations, and thus the challenge of providing solutions to these equations for a general overdensity distribution, $\delta = \frac{\delta\rho}{\bar{\rho}}$, is considerable. The subject of *Eulerian perturbation theory* has developed in response to this challenge, and is now a considerable area of study in its own right (aside from *Lagrangian perturbation theory*, which attacks the problem with a Lagrangian description of the fluid, leading to the famous "Zel'dovich Approximation" at first order [84].). Under the assumption of vanishing vorticity, we can define the velocity potential, $v$, such that $v_i = \partial_i v$, and the velocity divergence, $\theta = \partial^i v_i$. Neglecting pressure and substituting in for the Poisson equation, we obtain

$$\delta'_N + \theta_N = -\partial^i \left(\delta_N v_{Ni}\right) \tag{4.4}$$

$$\theta'_N + \mathcal{H}\theta_N + \frac{3\mathcal{H}^2}{2}\delta_N = -\partial^i \left(v_{Nj}\partial^j v_{Ni}\right) . \tag{4.5}$$

Eulerian perturbation theory is a method for providing approximate solutions to (4.4) and (4.5) in the *quasilinear* or *weakly nonlinear* regime, namely where the variance of fluctuations $\sigma^2(R) \lesssim 1$, for a given scale $R$. Here we have chosen to append the subscript "$N$" to variables, to remind the reader that these variables are Newtonian in nature, and we have implicitly assumed that the background is Einstein-de Sitter (although one can perform Newtonian perturbation theory on a ΛCDM background with modified time dependency factors). We will give an introduction to *Goroff's method* [85], which allows one to find expressions for the fastest growing mode solution in Einstein-de Sitter universes at *all orders* in terms of the *PT kernels*, integration kernels that encapsulate the mode-coupling behaviour induced by the nonlinearity of the equations [84]. Solutions involving transient modes can be identified using a Dyson series method developed by Scoccimarro [86], analogous to those used in quantum field theory; however, for our purposes Goroff's method will suffice. One recovers the PT kernels from Scoccimaro's method by neglecting the transient modes, and the equivalence of the results for the PT kernels extracted from both of these methods after suitable permutations is well established [87].

### 4.1.1. Finding solutions

Goroff's method proceeds as follows: first, one inserts the series decomposition

$$\delta_N = \delta_N^{(1)} + \frac{1}{2}\delta_N^{(2)} + \frac{1}{3!}\delta_N^{(3)} + \dots , \tag{4.6}$$

$$\theta_N = \theta_N^{(1)} + \frac{1}{2}\theta_N^{(2)} + \frac{1}{3!}\theta_N^{(3)} + \dots , \tag{4.7}$$

into the fully nonlinear equations, (4.4) and (4.5). Discarding all subleading order terms and quadratic and higher order products, one then solves the linearised equations for the leading order terms in the series (7.26). This is equivalent to making the critical assumptions that $\delta_N^{(1)}\theta_N^{(1)} \ll \delta_N^{(1)\prime}$ and $(\theta_N^{(1)})^2 \ll \theta_N^{(1)\prime}$. The linearised equations for the leading order terms in Fourier space then reduce to

$$\delta_N^{(1)''}(\boldsymbol{k},\tau) + \mathcal{H}\delta_N^{(1)'}(\boldsymbol{k},\tau) - \frac{3}{2}\mathcal{H}^2\delta_N^{(1)}(\boldsymbol{k},\tau) = 0 \ . \tag{4.8}$$

Performing a separation of variables, $\delta_N^{(1)}(\boldsymbol{k},\tau) = \delta^{(1)}(\boldsymbol{k})\mathcal{D}(\tau)$, and discarding decaying solutions, one finds that the linear solution for the leading order terms can be described by a spatial initial condition, $\delta^{(1)}(\boldsymbol{k})$, which we take to be a Gaussian random field - as predicted by a large *CLASS* of inflationary models, and the growth factor, $\mathcal{D}$ , which has the convenient linear form in EdS, $\mathcal{D}(\tau) = a$. The leading order linear velocity divergence can be obtained in terms of this initial condition from the linearised continuity equation,

$$\delta_N^{(1)'} + \theta_N^{(1)} = 0 \qquad \Longrightarrow \qquad \theta_N^{(1)} = -\mathcal{H}(\tau)a(\tau)\delta^{(1)}(\boldsymbol{k}) \ , \tag{4.9}$$

and the leading order linear gravitational potential can be found using the Poisson equation,

$$\nabla^2 U^{(1)} = \frac{3\mathcal{H}^2}{2}\delta_N^{(1)} \qquad \Longrightarrow \qquad U^{(1)}(\boldsymbol{k}) = \frac{-3\mathcal{H}^2 a}{2k^2}\delta^{(1)}(\boldsymbol{k}) = \varphi(\boldsymbol{k}) \ , \tag{4.10}$$

where we note that since the combination $\mathcal{H}^2 a = 4$ in Einstein-de Sitter, the gravitational potential is also time independent, and we introduce the notation $\varphi$ for the initial condition in terms of the gravitational potential (which will be useful when we discuss relativistic perturbation theory, since in longitudinal gauge, the leading order density contrast is no longer separable).

The full evolution of the first order perturbations, $\delta_N$, $\theta_N$ and $U$ is then completely specified. We have successfully constructed our first inhomogeneous model universe - an Einstein-de Sitter background, superimposed with linear Newtonian scalar fluctuations describing the evolution of inhomogeneities in the density contrast, Newtonian gravitational potential and peculiar velocity. The statistical properties of the initial condition $\delta^{(1)}$ are in general predicted by an inflationary model, which completes the description. Whilst this model is not sufficient to describe the real universe due to the existence of the cosmological constant, the complex behaviour of coupled radiation fluids in the early universe, and the growth of nonlinear structure in the late universe, it is nevertheless extremely useful as a starting point from which to build more complex models.

If a more accurate approximation to the full nonlinear Newtonian dynamics is required, one can proceed with higher-order perturbation theory. Having obtained the linear solutions, one can then substitute these back into the original equations. Cancelling away the linear terms, and neglecting third order quantities and products, one is left with a set of inhomogeneous ordinary differential equations (ODEs) for the second order terms, where the inhomogeneous source terms are constructed from quadratic products of the linear leading order solutions. To find a solution for the second order quantities, all that is required is to find a particular solution for the given source terms. One can repeat this process iteratively to find successively higher order corrections, each hopefully providing a better approximation to the true quantities than the last. Given that the linear growth factor $\mathcal{D} \sim a$ in Einstein-de Sitter, it is not hard to verify that $a^2$ will be a particular solution for the time dependency of the second order equations, and that $a^3$ will be a particular solution to the third order equations etc. All that is left then, is to determine the spatial dependencies for each order, which correspond to the mode coupling. It is precisely this information that is encoded in the PT kernels. This motivates us to adopt the following split for our perturbations:

$$\delta_N(\boldsymbol{k},\tau) = \sum_{n=1}^{\infty} \frac{\delta_N^{(n)}(\boldsymbol{k})}{n!}\, a^n \tag{4.11}$$

$$\theta_N(\boldsymbol{k},\tau) = -\mathcal{H}\sum_{n=1}^{\infty} \frac{\theta_N^{(n)}(\boldsymbol{k})}{n!}\, a^n \ , \tag{4.12}$$

where

$$\delta_N^{(n)}(\boldsymbol{k}) = \int \left( \prod_{i=1}^{n} \frac{\mathrm{d}^3 q_i}{(2\pi)^{3i}} \delta^{(1)}(\boldsymbol{q}_i) \right) (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \sum_{i=1}^{n} \boldsymbol{q}_i) F_n(\boldsymbol{q}_1, \boldsymbol{q}_2, ..., \boldsymbol{q}_n) \ , \tag{4.13}$$

$$\theta_N^{(n)}(\boldsymbol{k}) = \int \left( \prod_{i=1}^{n} \frac{\mathrm{d}^3 q_i}{(2\pi)^{3i}} \delta^{(1)}(\boldsymbol{q}_i) \right) (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \sum_{i=1}^{n} \boldsymbol{q}_i) G_n(\boldsymbol{q}_1, \boldsymbol{q}_2, ..., \boldsymbol{q}_n) \ , \tag{4.14}$$

and $F_n$ and $G_n$ are the PT kernels. Determining these kernels as functions of the internal momenta then constitutes the constructions of an all orders solution to Equations (4.4) and (4.5).

In order to determine the perturbation theory kernels, we Fourier transform the nonlinear equations to get

$$\delta'_{\mathrm{N}}(\boldsymbol{k},\tau) + \theta_{\mathrm{N}}(\boldsymbol{k},\tau) = -\int \frac{\mathrm{d}^3p_1\,\mathrm{d}^3p_2}{(2\pi)^3}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}_1-\boldsymbol{p}_2)\times \left[\alpha(\boldsymbol{p}_1,\boldsymbol{p}_2)\,\delta_{\mathrm{N}}(\boldsymbol{p}_1,\tau)\,\theta_{\mathrm{N}}(\boldsymbol{p}_2,\tau)\right], \tag{4.15}$$

$$\theta'_{\mathrm{N}}(\boldsymbol{k},\tau) + \mathcal{H}\theta_{\mathrm{N}}(\boldsymbol{k},\tau) + \frac{3}{2}\mathcal{H}^2\delta_{\mathrm{N}}(\boldsymbol{k},\tau) = -\int \frac{\mathrm{d}^3p_1\,\mathrm{d}^3p_2}{(2\pi)^3}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}_1-\boldsymbol{p}_2)\times \left[\beta(\boldsymbol{p}_1,\boldsymbol{p}_2)\,\theta_{\mathrm{N}}(\boldsymbol{p}_1,\tau)\,\theta^{(1)}_{\mathrm{N}}(\boldsymbol{p}_2,\tau)\right], \tag{4.16}$$

where $\alpha$ and $\beta$ are given by

$$\alpha(\boldsymbol{p}_1,\boldsymbol{p}_2) \equiv \frac{\boldsymbol{k}\cdot\boldsymbol{p}_2}{p_2^2} \quad \text{and} \quad \beta(\boldsymbol{p}_1,\boldsymbol{p}_2) \equiv \frac{k^2\,\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{2\,p_1^2\,p_2^2}\,, \tag{4.17}$$

with $\boldsymbol{k} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ enforced by the Dirac delta function. The quantities $\alpha$ and $\beta$ are referred to as the *vertex couplings* by analogy with quantum field theory. We can now insert our perturbation series in the form of Eqs. (4.11) and (4.12) into Eqs. (4.15) and (4.16). This results in the following (somewhat cumbersome) expressions for the continuity equation:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{a^n\mathcal{H}}{n!}\Bigg\{\int\left(\prod_{i=1}^{n}\frac{\mathrm{d}^3k_i}{(2\pi)^{3i}}\delta^{(1)}(\boldsymbol{k}_i)\right)(2\pi)^3\delta^{(3)}\left(\boldsymbol{k}-\sum_{i=1}^{n}\boldsymbol{k}_i\right)\times \\
&\qquad\left[nF_n(\boldsymbol{k}_1,\boldsymbol{k}_2,...,\boldsymbol{k}_n) - G_n(\boldsymbol{k}_1,\boldsymbol{k}_2,...,\boldsymbol{k}_n)\right]\Bigg\} \\
&=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a^m a^l\mathcal{H}}{m!l!}\Bigg\{\int\frac{\mathrm{d}^3p_1\,\mathrm{d}^3p_2}{(2\pi)^3}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}_1-\boldsymbol{p}_2) \\
&\quad\times\left(\prod_{j=1}^{m}\frac{\mathrm{d}^3q_{1j}}{(2\pi)^{3j}}\delta^{(1)}(\boldsymbol{q}_{1j})\right)\left(\prod_{k=1}^{l}\frac{\mathrm{d}^3q_{2k}}{(2\pi)^{3k}}\delta^{(1)}(\boldsymbol{q}_{2k})\right)(2\pi)^6\delta^{(3)}\left(\boldsymbol{p}_1-\sum_{j=1}^{m}\boldsymbol{q}_{1j}\right) \\
&\quad\times\delta^{(3)}\left(\boldsymbol{p}_2-\sum_{k=1}^{l}\boldsymbol{q}_{2k}\right)F_m(\boldsymbol{q}_{11},\boldsymbol{q}_{12},...,\boldsymbol{q}_{1m})G_l(\boldsymbol{q}_{21},\boldsymbol{q}_{22},...,\boldsymbol{q}_{2l})\,\alpha(\boldsymbol{p}_1,\boldsymbol{p}_2)\Bigg\}\,,
\end{aligned} \tag{4.18}$$

and the Euler equation:

$$\sum_{n=1}^{\infty} \frac{a^n \mathcal{H}}{n!} \Bigg\{ \int \left( \prod_{i=1}^{n} \frac{\mathrm{d}^3 k_i}{(2\pi)^{3i}} \delta^{(1)}(\boldsymbol{k}_i) \right) (2\pi)^3 \delta^{(3)} \left( \boldsymbol{k} - \sum_{i=1}^{n} \boldsymbol{k}_i \right) \times$$
$$\left[ (2n+1) G_n(\boldsymbol{k}_1, \boldsymbol{k}_2, ..., \boldsymbol{k}_n) - 3F_n(\boldsymbol{k}_1, \boldsymbol{k}_2, ..., \boldsymbol{k}_n) \right] \Bigg\}$$
$$= \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{a^m a^l \mathcal{H}}{m! l!} \Bigg\{ \int \frac{\mathrm{d}^3 p_1 \, \mathrm{d}^3 p_2}{(2\pi)^3} \, \delta^{(3)}(\boldsymbol{k} - \boldsymbol{p}_1 - \boldsymbol{p}_2)$$
$$\times \left( \prod_{j=1}^{m} \frac{\mathrm{d}^3 q_{1j}}{(2\pi)^{3j}} \delta^{(1)}(\boldsymbol{q}_{1j}) \right) \left( \prod_{k=1}^{l} \frac{\mathrm{d}^3 q_{2k}}{(2\pi)^{3k}} \delta^{(1)}(\boldsymbol{q}_{2k}) \right) (2\pi)^6 \delta^{(3)} \left( \boldsymbol{p}_1 - \sum_{j=1}^{m} \boldsymbol{q}_{1j} \right)$$
$$\times \, \delta^{(3)} \left( \boldsymbol{p}_2 - \sum_{k=1}^{l} \boldsymbol{q}_{2k} \right) G_m(\boldsymbol{q}_{11}, \boldsymbol{q}_{12}, ..., \boldsymbol{q}_{1m}) G_l(\boldsymbol{q}_{21}, \boldsymbol{q}_{22}, ..., \boldsymbol{q}_{2l}) \; \beta(\boldsymbol{p}_1, \boldsymbol{p}_2) \Bigg\} \; . \quad (4.19)$$

Evaluating the integrals over $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, and selecting the $n^{\text{th}}$ term from each expression, it is easy to see that by relabelling the integration variables $\boldsymbol{q}_{1j}$ and $\boldsymbol{q}_{2k}$ as $\boldsymbol{k}_i$, that one can equate the two integrands, and therefore be left with the following purely algebraic expressions for the $n^{\text{th}}$-order kernels in terms of products of lower-order kernels:

$$nF_n(\boldsymbol{k}_{1...n}) - G_n(\boldsymbol{k}_{1...n}) = \sum_{m=1}^{m=n-1} \frac{n!}{m!(n-m)!} \Big[ \alpha(\boldsymbol{k}_{1:m}, \boldsymbol{k}_{m:n})$$
$$\times \, F_m(\boldsymbol{k}_{1...m}) G_{n-m}(\boldsymbol{k}_{m...n}) \Big] \; , \quad (4.20)$$

$$(2n+1) G_n n(\boldsymbol{k}_{1...n}) - 3F_n(\boldsymbol{k}_{1...n}) = \sum_{m=1}^{m=n-1} \frac{n!}{m!(n-m)!} \Big[ 2\beta(\boldsymbol{k}_{1:m}, \boldsymbol{k}_{m:n})$$
$$\times \, G_m(\boldsymbol{k}_{1...m}) G_{n-m}(\boldsymbol{k}_{m...n}) \Big] \; , \quad (4.21)$$

where we have used the shorthand notation $F_n(\boldsymbol{k}_{1...n}) = F_n(\boldsymbol{k}_1, \boldsymbol{k}_2, ..., \boldsymbol{k}_n)$ and $\boldsymbol{k}_{i:j} = \boldsymbol{k}_i + \boldsymbol{k}_{i+1} + \cdots + \boldsymbol{k}_{j-1} + \boldsymbol{k}_j$.

It is important to note that we are free to relabel the integration variables in any manner we choose. This implies that we should symmetrise on the wavevectors $\boldsymbol{k}_i$ since each permutation corresponds to a different relabelling of the integration variables, all of which are equivalent. Generally, it is easiest to perform this procedure at the end of the calculation, so we will leave it until then. It is easy to solve these algebraic equations for $F_n$ and $G_n$ (the unsymmetrised kernels). The resulting expressions are

$$F_n(\boldsymbol{k}_{1\dots n}) = \sum_{m=1}^{m=n-1} \binom{n}{m} \frac{G_{n-m}(\boldsymbol{k}_{m\dots n})}{(2n+3)(n-1)} \Big\{ (2n+1)\alpha(\boldsymbol{k}_{1:m}, \boldsymbol{k}_{m:n}) F_m(\boldsymbol{k}_{1\dots m}) + 2\beta(\boldsymbol{k}_{1:m}, \boldsymbol{k}_{m:n}) G_m(\boldsymbol{k}_{1\dots m}) \Big\} , \quad (4.22)$$

$$G_n(\boldsymbol{k}_{1\dots n}) = \sum_{m=1}^{m=n-1} \binom{n}{m} \frac{G_{n-m}(\boldsymbol{k}_{m\dots n})}{(2n+3)(n-1)} \Big\{ 3\alpha(\boldsymbol{k}_{1:m}, \boldsymbol{k}_{m:n}) F_m(\boldsymbol{k}_{1\dots m}) + 2n\beta(\boldsymbol{k}_{1:m}, \boldsymbol{k}_{m:n}) G_m(\boldsymbol{k}_{1\dots m}) \Big\} . \quad (4.23)$$

The reader will notice an additional factor of $\binom{n}{m}$ compared to the standard expressions in the literature. These factors come from our choice to include factors on $\frac{1}{n!}$ in the perturbation expansion, so as to match up with the expansions in traditional relativistic perturbation theory. This normalisation choice is purely arbitrary and has no effect on the physics.

Given that $F_1 = G_1 = 1$ by definition, one is able to recursively calculate all the $F_n$ and $G_n$ using the above relations and symmetrise to obtain the final PT kernels. For example, consider the $n = 2$ case:

$$F_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{2}{7}\big(5\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + 2\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)\big) , \quad (4.24)$$

which upon symmetrising yields the familiar expression,

$$F_2^{(s)}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{10}{7} + \frac{4}{7}(\hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2)^2 + \hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2 \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) . \quad (4.25)$$

Similarly, the expression for $G_2^{(s)}$ is

$$G_2^{(s)}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{6}{7} + \frac{8}{7}(\hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2)^2 + \hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2 \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) . \quad (4.26)$$

The integral kernels have the following properties:

- As the sum $\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2 \dots \boldsymbol{k}_n$ vanishes, but individual $\boldsymbol{k}_i$ do not, $K_n \propto k^2$ as is required by momentum conservation in the centre of mass frame.
- If some of the $\boldsymbol{k}_i$ become large, but the total sum $\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2 \dots \boldsymbol{k}_n$ remains fixed, the kernel vanishes in the inverse square law, i.e.

$K_n(\boldsymbol{k}_1, \boldsymbol{k}_2, \dots \boldsymbol{k}_{n-2}, \boldsymbol{p}, -\boldsymbol{p}) \propto \frac{k^2}{p^2}$ for $p \gg q_i$.

- If any of the individual $\boldsymbol{k}_i \to 0$, an infrared divergence with the structure $\frac{\boldsymbol{k}_i}{k^2}$ in the coupling vertices $\alpha$ and $\beta$ is induced in the integral kernels. There are no infrared divergences as partial sums of several wavevectors go to zero.

Expressions up to $F_3^{(s)}$ and $G_3^{(s)}$ are well known and can be found in [85]. Unsymmetrised expressions for $F_4$ and $G_4$ have been calculated, but it is unfeasible to continue beyond this point due to the rapid increase in the number of terms ($F_4$ and $G_4$ have 8523 terms each).

## 4.2. Cosmological perturbation theory

In the previous section, we showed how to solve the simplified problem of the evolution of perturbations in an Einstein-de Sitter universe, governed by Newtonian gravity. The relativistic version of this problem in ΛCDM universes will be the subject of this section. These techniques have been described in many other works; we direct the interested reader to [88] and the appendices of [89] for a full description of second order perturbations in a ΛCDM universe. For general questions about cosmological perturbation theory, see the reviews by Malik and Wands, [31], Mukhanov, Feldman and Brandenburger, [90], or Kodoma and Sasaki, [91].

*Cosmological perturbation theory* (as the relativistic perturbation theory is commonly known) begins with the fundamental assumption that *all fluctuations are small*. That is to say, we assume

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \ , \tag{4.27}$$

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \ , \tag{4.28}$$

where

$$\delta g_{\mu\nu} \sim \delta \sim v \sim \epsilon \sim 10^{-4} \ . \tag{4.29}$$

Here, we introduce $\epsilon$ as a smallness parameter, similar to the way that we used $\eta$ to keep track of the sizes of quantities in post-Newtonian gravity. The size of epsilon is set by the typical depth of gravitational potential wells in the real universe.

The reader should recognise that this expansion satisfies the conditions of being a *weak field expansion*, but that it is a fundamentally different type of weak field expansion to post-Newtonian gravity. Namely, in post-Newtonian gravity, we found that the typical relationship between the leading order Newtonian gravitational field $U$ and the peculiar velocity $v$ satisfied $U \sim v^2 \sim \eta^2$, as is typical in the collapsed virialised systems that are well modelled by post-Newtonian expansions. In contrast, in cosmological perturbation theory (or "CPT"), the typical relationship between gravitational potentials and peculiar velocities is $\phi \sim v \sim \epsilon$. This should already indicate that the nature of the expansion is fundamentally different. Furthermore, there has been no assumption so far about scale of applicability - unlike in post-Newtonian gravity, which fundamentally came with a built-in restriction to the near-zone. In principle, CPT is applicable to *any scale*, so long as the relationship in Equation (4.29) holds. In practice, when dealing with matter fluctuations, these conditions are typically expected to break down for length scales below $\sim 100\,\mathrm{Mpc}$ in the late universe.

We can irreducibly decompose our metric perturbations as before,

$$\delta g_{00} = -2a^2\phi \tag{4.30}$$

$$\delta g_{0i} = a^2(B_{,i} - S_i) \tag{4.31}$$

$$\delta g_{ij} = a^2(-2\psi\,\delta_{ij} + 2E_{,ij} + 2F_{(i,j)} + h_{ij})\,, \tag{4.32}$$

whilst the components of the perfect fluid stress energy tensor up to first order are given by

$$T^0{}_0 = \bar{T}^0{}_0 + \delta T^0{}_0 = -(\bar{\rho} + \bar{\rho}\delta) \tag{4.33}$$

$$T^0{}_i = \bar{T}^0{}_i + \delta T^0{}_i = (\bar{\rho} + \bar{P})(v_i + B_{,i} - S_i) \tag{4.34}$$

$$T^i{}_j = \bar{T}^i{}_j + \delta T^i{}_j = (\bar{P} + \delta P)\delta^i{}_j\,. \tag{4.35}$$

Another point of contrast to post-Newtonian gravity is the size of $\delta = \delta\rho/\bar{\rho} \sim \epsilon$. In post-Newtonian gravity, we simply separated out the leading order energy density into a homogeneous part, $\bar{\rho}_N \sim \eta^2\, L_N^{-2}$, and an inhomogeneous part, $\delta\rho_N \sim \eta^2\, L_N^{-2}$, such that when we take the ratio, $\delta_N = \delta\rho_N/\bar{\rho}_N \sim 1$.

### 4.2.1. Gauge problem and gauge invariant quantities

The gauge problem in cosmological perturbation theory has a slightly different structure to the gauge problem in post-Newtonian gravity. Whilst the reasons for its existence are the same, namely the non-uniqueness of mapping between perturbed and unperturbed spacetimes, the expansion scheme works in a different way, resulting in the gauge generators having different sizes compared to the gauge generators in post-Newtonian gravity [80]. This results in a different set of gauge transformations describing a different residual coordinate freedom.

Recall the fundamental gauge transformation:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \mathcal{L}_{\xi}\, g_{\mu\nu} + \frac{1}{2}\mathcal{L}^2_{\xi}\, g_{\mu\nu} + \,\dots\,, \tag{4.36}$$

which gives the various components of the metric transforming as

$$g_{00} \rightarrow g_{00} + \xi^{\mu}\partial_{\mu} g_{00} + 2g_{0\mu}\,\xi^{\mu\prime} + \dots \tag{4.37}$$

$$g_{0i} \rightarrow g_{0i} + \xi^{\mu}\partial_{\mu} g_{0i} + g_{0\mu}\,\xi^{\mu}_{,i} + g_{i\mu}\,\xi^{\mu\prime} + \dots \tag{4.38}$$

$$g_{ij} \rightarrow g_{ij} + \xi^{\mu}\partial_{\mu} g_{ij} + 2g_{\mu(i}\,\xi^{\mu}{}_{,j)} + \dots \tag{4.39}$$

where the ellipses in these expression denote terms that are quadratic or higher-order in the gauge generator, $\xi^{\mu}$. It can immediately be seen from these equations that every component of the gauge generators must be of the same order-of-magnitude in the perturbative expansion as the metric perturbations, i.e. that

$$\xi^{\mu} \sim \epsilon\,. \tag{4.40}$$

If $\xi^{\mu}$ were larger than this, then the metric (after the gauge transformation), could no longer be written as perturbed FLRW.

Using this information, and the decomposition given in Equation (4.30), we then find the standard set of gauge transformations:

$$\begin{aligned}
\phi &\rightarrow \phi + \mathcal{H}\,\xi^0 + \xi^{0\prime} \\
B &\rightarrow B + \zeta' - \xi^0 \\
S_i &\rightarrow S_i - \zeta'_i \\
\psi &\rightarrow \psi - \mathcal{H}\,\xi^0 \\
E &\rightarrow E + \zeta \\
F_i &\rightarrow F_i + \zeta_i \\
h_{ij} &\rightarrow h_{ij}\,,
\end{aligned} \tag{4.41}$$

where we have decomposed the spatial part of the gauge generator, such that $\xi_i = \zeta_{,i} + \zeta_i$, where $\zeta^i$ is divergenceless. It can be seen that *all* metric perturbations transform under a general gauge transformation, with the notable exception of $h_{ij}$ (at linear order). This situation is to be contrasted with that in post-Newtonian theory, where the leading order gravitational potential $U$ was gauge invariant.

Similarly, we can calculate how the components of the stress-energy tensor transform under a gauge transformation with $\xi^\mu \sim \epsilon$. For a perfect fluid, these components can be written

$$T^0{}_0 = -(\bar{\rho} + \delta\rho) \tag{4.42}$$

$$T^0{}_i = (\bar{\rho} + \bar{p})(v_i + B_{,i} - S_i) \tag{4.43}$$

$$T^i{}_j = (\bar{p} + \delta p)\delta^i{}_j \tag{4.44}$$

and under the transformation (6.26) therefore give

$$\begin{aligned} \delta &\to \delta + \xi^0 \frac{\bar{\rho}'}{\bar{\rho}} \\ \delta p &\to \delta p + \xi^0 \bar{p}' \\ v^i &\to v^i - \xi^{i\prime} . \end{aligned} \tag{4.45}$$

It is again apparent that all perturbed matter quantities transform under a general gauge transformation, again in sharp contrast to the situation in post-Newtonian gravity, where the leading order matter variables were gauge invariant.

It is easy to see that we can define the following gauge invariant quantities:

$$\Phi = \phi + \mathcal{H}(B - E') + (B - E')' \, , \tag{4.46}$$

$$\Psi = \psi - \mathcal{H}(B - E') \, . \tag{4.47}$$

These potentials are known as the Bardeen potentials, and were derived in Bardeen's seminal paper on the subject [32].

From this point onwards, we will choose to work primarily in conformal Newtonian gauge (sometimes referred to as Poisson gauge), which will again be defined by the easily achievable gauge conditions

$$B = E = F_i = 0 \, . \tag{4.48}$$

In this gauge, it can be seen that the diagonal scalar components of the metric are simply equal to the gauge invariant Bardeen potentials, a very useful property. Many other viable gauges exist in cosmological perturbation theory, however the Poisson gauge has the property of being realisable in both post-Newtonian expansions and cosmological perturbation theory, making it uniquely useful for our purposes. We will come to discuss this issue in greater detail in a coming chapter.

### 4.2.2. Scalar Solutions in Poisson gauge

We will give a treatment of the method used to find solutions to the scalar cosmological perturbation theory equations in the Poisson gauge, since the subsequent methods used are similar to these, and the solution of this system will provide a good reference point against which to compare future results. We will focus on the density contrast, since this is the object we will calculate in the two-parameter perturbation theory we come to develop in subsequent sections. These methods are well known and are discussed in detail in [88] and [89].

#### Linear order

It is a well known fact that scalar, vector and tensor degrees of freedom decouple at linear order in perturbation theory. The scalar modes are responsible for the growth of structure in our universe, with tensor and vector modes decaying very rapidly in FLRW universes, for typical inflationary initial conditions [1]. Given the metric and stress energy tensor components above, the first order scalar Einstein equations in Poisson gauge for a perfect fluid stress-energy tensor are then given by

$$\Psi_1'' + 3\mathcal{H}\Psi_1' + \Lambda a^2\Phi_1 = \mathcal{H}(\Psi_1' - \Phi_1') + \frac{1}{3}\nabla^2(\Psi_1 - \Phi_1) \;, \tag{4.49}$$

$$\partial^i\partial_j(\Psi_1 - \Phi_1) = \frac{1}{3}\delta^i_j\,\nabla^2(\Psi_1 - \Phi_1) \;, \tag{4.50}$$

$$\frac{1}{3}\nabla^2\Psi_1 - \mathcal{H}\Psi_1' - \mathcal{H}^2\Phi_1 = \frac{4\pi a^2\bar{\rho}}{3}\,\delta_1 \;, \tag{4.51}$$

$$\partial_i\big(\Psi_1' + \mathcal{H}\Phi_1\big) + \frac{3\mathcal{H}^2}{2}v_{1i} = 0 \;\;, \tag{4.52}$$

where we have appended the subscript “1” in order to remind the reader that we are dealing with the linear order approximations to these quantities. We take note of the fact that we now have four scalar quantities, $\{\delta_1, v_1, \Phi_1, \Psi_1\}$, where $\partial_i v_1 = v_{1i}$ defines the velocity potential, and four equations, three of which are constraints and one of which (Equation (4.49)) is is an evolution equation. We can therefore describe the evolution of this system in terms of a single scalar degree of freedom. Since the Einstein equations form a closed set, there is no need to appeal to conservation equations to close the system as was done in Newtonian perturbation theory. We

[1]More esoteric sources of perturbations, such as cosmic strings, can generate larger amounts of tensor and vector modes [92].

therefore choose to work directly with the gravitational potentials as our scalar degree of freedom, although this choice is not unique.

This system is easily solved by noting that Equation (4.50) forces $\Phi_1 = \Psi_1 = \varphi(x, \tau)$. The time evolution of the system is then described by a single degree of freedom, and Equation (4.49) reduces to

$$\varphi'' + 3\mathcal{H}\varphi' + \Lambda a^2 \varphi = 0 \; . \tag{4.53}$$

We can solve this system by separating variables, introducing $\varphi(x, \tau) = \varphi_0(x) g(\tau)$, such that $\mathcal{D}(a) = a(\tau) g(\tau)$, where $\mathcal{D}$ is the *linear growth factor*, $a$ is the scale factor, and $g(\tau)$ is the *growth suppression factor*. In our exposition on Newtonian perturbation theory, we found that $\mathcal{D}(a) = a$ in Einstein-de Sitter universes. In the more general $\Lambda$CDM scenario, we find that $g(\tau)$ and $\mathcal{D}(a)$ can be defined as solutions to two related ordinary differential equations:

$$g'' + 3\mathcal{H}g' + a^2 \Lambda g = 0 \; , \tag{4.54}$$

which implies

$$\mathcal{D}'' + \mathcal{H}\mathcal{D}' - \frac{3\mathcal{H}_0^2 \Omega_{m0}}{2a} \mathcal{D} = 0 \; . \tag{4.55}$$

The second of these reduces to the same differential equation that governed the growth of the density contrast in Newtonian perturbation theory in the Einstein-de Sitter limit, prompting us to associate $\varphi$ with the Newtonian gravitational potential. Equation (4.55) has two solutions,

$$\mathcal{D}_- \propto \frac{\mathcal{H}}{a} \; , \tag{4.56}$$

$$\mathcal{D}_+ = \frac{5}{2} \Omega_{m0} \mathcal{H}_0^2 \left( \frac{\mathcal{H}}{a} \right) \int_0^a \frac{\mathrm{d}\tilde{a}}{\mathcal{H}^3(\tilde{a})} \; , \tag{4.57}$$

$$= a \sqrt{1 + \frac{\Omega_{\Lambda 0}}{\Omega_{m0}} a^3} \; {}_2F_1 \left( \frac{3}{2}, \frac{5}{6}, \frac{11}{6}, -\frac{\Omega_{\Lambda 0}}{\Omega_{m0}} a^3 \right) \; , \tag{4.58}$$

where ${}_2F_1$ is the hypergeometric function. The first of these solutions represents a decaying mode, whilst the second represents a growing solution. For our purposes, it will be sufficient to neglect the effects of the decaying mode, although we will eventually have to consider its effects later on. We will choose to normalise the growth rate such that $\mathcal{D}(\tau_0) = 1$. We therefore have $\mathcal{D}(\tau) = \mathcal{D}_+(\tau)/\mathcal{D}_+(\tau_0)$. The growth factors from $\Lambda$CDM and Einstein-de Sitter cosmologies are plotted in Figure

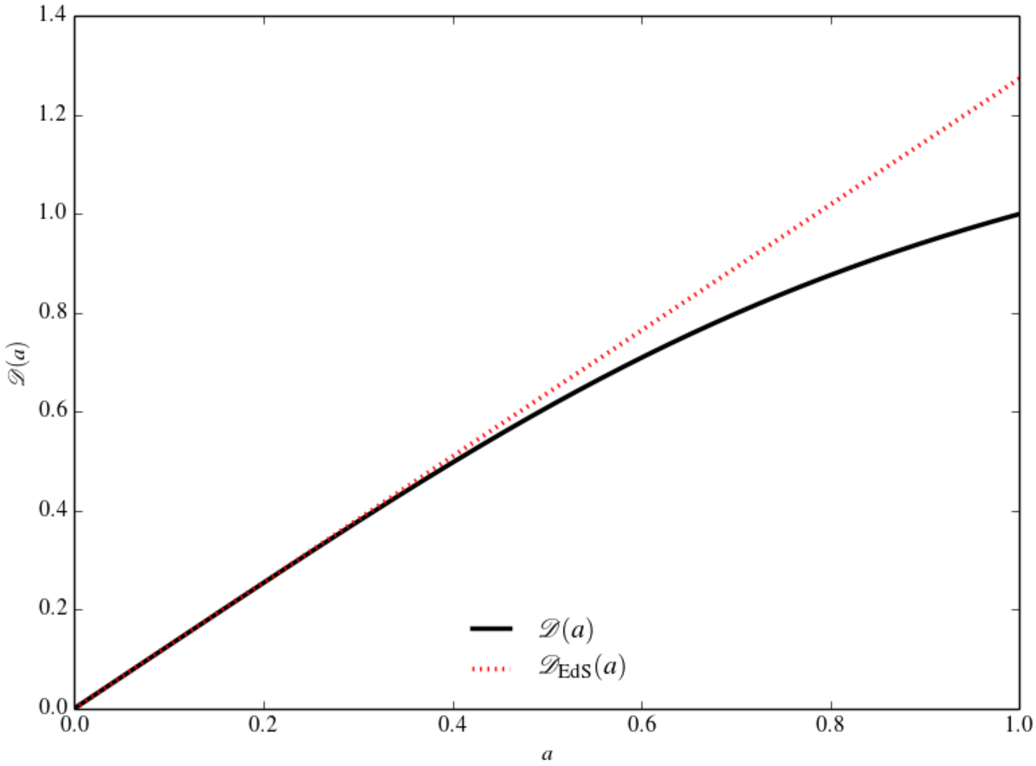


Figure 4.1.: A plot of the growth factor $\mathcal{D}(a)$ in $\Lambda$CDM and EdS. Here $\mathcal{D}_{\rm EdS} = a/\mathcal{D}_{\Lambda\rm CDM}(a=1)$ in order to account for the normalisation of the $\Lambda$CDM growth factor, and the enhanced growth that occurs in EdS.

4.1. We have chosen to normalise the Einstein-de Sitter growth rate by a factor of $\mathcal{D}(a=1)$ as well in order to account for the corresponding normalisation that occurs in the $\Lambda$CDM cosmology. This is important, since we wish to compare results in the two cosmologies.

There are a number of useful identities relating $g$, $\mathcal{D}$, and another quantity $f$, known as the *growth rate of structure*, defined as

$$\frac{\mathrm{d}\log\mathcal{D}}{\mathrm{d}\log a} = f(\Omega_m) \ . \tag{4.59}$$

We give a non-exhaustive list of these identities here:

$$\mathcal{D}' = f\mathcal{H}\mathcal{D} \ , \tag{4.60}$$

$$g' = \mathcal{H}g(f-1) \tag{4.61}$$

$$\frac{\mathcal{H}\mathcal{D}'}{\mathcal{H}_0^2\Omega_{m0}} = \frac{5}{2}g_{\rm in} - \frac{3}{2}g \ , \tag{4.62}$$

where the subscript "in" stands for "initial". A derivation of the last of these

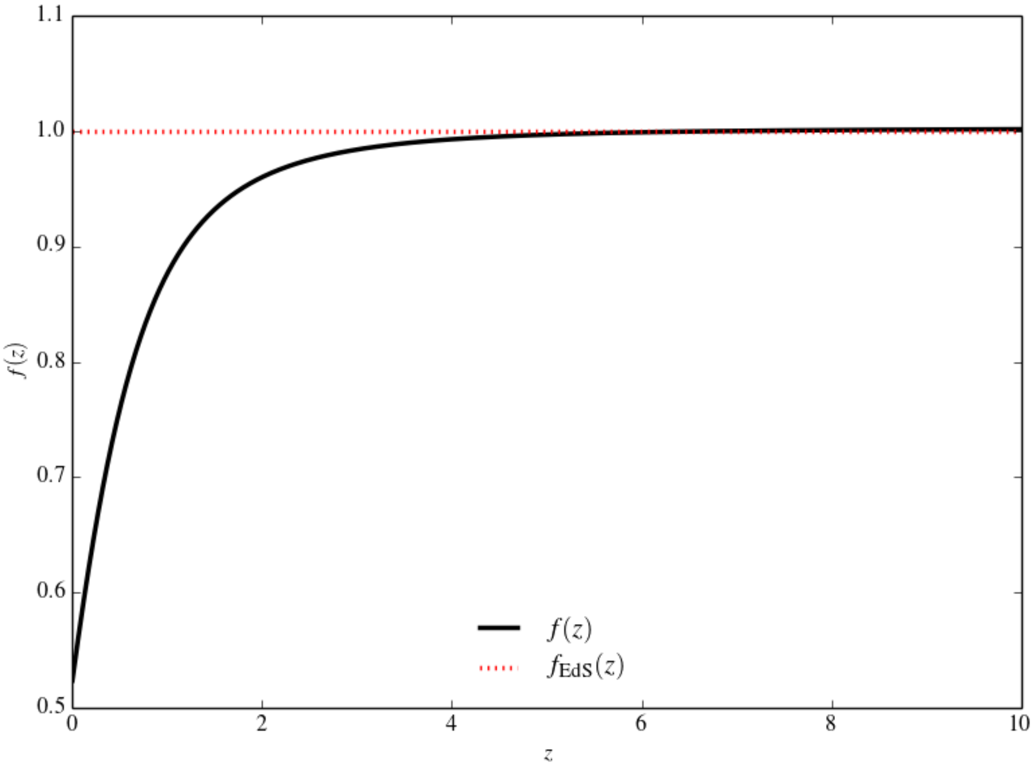


Figure 4.2.: A plot of the growth rate $f(z)$ in ΛCDM. Here $f_{\rm EdS} = 1$ by definition, so we see that $f(z)$ measures how growth of structure is generically suppressed by the inclusion of the cosmological constant. Deviation from matter domination becomes significant around $z \sim 6$. A knock-on implication of this is that structures have to form earlier in ΛCDM cosmologies in order to be consistent with the data.

identities is given in the paper [88], and $g_{\rm in}$ is defined by $\varphi_0(\boldsymbol{x})g_{\rm in} = \varphi(\boldsymbol{x}, \tau_{\rm in})$, and can be easily shown to be a constant.

We will choose to ignore the decaying solution, as is done in the majority of the literature. Expressions for all the other variables can then be calculated in terms of the initial condition for the potential, $\varphi_0$, using the constraint equations. We obtain the following expressions for the first order density contrast and velocity potential,

$$\delta_1 = \frac{2}{3\mathcal{H}_0^2\Omega_{m0}}(\mathcal{D}\nabla^2\varphi_0 - 3\mathcal{H}\mathcal{D}'\varphi_0) \ , \tag{4.63}$$

$$v_1 = -\frac{2}{3\mathcal{H}_0^2\Omega_{m0}}\mathcal{D}'\varphi_0 \ , \tag{4.64}$$

$$\text{which implies} \quad \theta_1 = -\frac{2}{3\mathcal{H}_0^2\Omega_0}\mathcal{D}'\nabla^2\varphi_0 \quad . \tag{4.65}$$

In Fourier space, these expressions become

$$\delta_1(\boldsymbol{k},\tau) = -\frac{2k^2}{3\mathcal{H}_0^2\Omega_{m0}}\mathcal{D}\left(1+\frac{3f\mathcal{H}^2}{k^2}\right)\varphi_0(\boldsymbol{k}) \ , \tag{4.66}$$

$$v_1(\boldsymbol{k},\tau) = -\frac{2}{3\mathcal{H}_0^2\Omega_{m0}}\mathcal{D}'\varphi_0(\boldsymbol{k}) \ , \tag{4.67}$$

$$\text{which implies}\ \ \theta_1(\boldsymbol{k},\tau) = \frac{2k^2}{3\mathcal{H}_0^2\Omega_{m0}}\mathcal{D}'\varphi_0(\boldsymbol{k}) \ . \tag{4.68}$$

It is clear that the density contrast is not separable in this gauge. An effect of this is that it is often more practical to work directly with the gravitational potential, $\varphi$, rather than the density contrast as we did in Newtonian perturbation theory. The first term is the same as the the first-order Newtonian density contrast, whereas the second term is a GR correction that is only relevant at the largest scales, and is only present in this particular gauge. This correction is the result of the relative velocity between the Eulerian and Lagrangian frames leading to a different time coordinate in each frame.

### 4.2.3. Second order

The conformal Poisson gauge line element up to second order, neglecting vectors and tensors, can be written

$$ds^2 = a(\tau)^2\left[-\big(1+\varphi+\frac{1}{2}\Phi_2\big)+\delta_{ij}(1-\varphi-\frac{1}{2}\Psi_2\big)dx^i dx^j\right] \ , \tag{4.69}$$

whilst the components of the stress energy tensor in Poisson gauge are given by

$$T^0_{\ 0} = -\bar{\rho}\left[1+\delta_1+\frac{1}{2}\delta_2+v_1^2\right] \ , \tag{4.70}$$

$$T^i_{\ 0} = -\bar{\rho}\left[v_1^i+\frac{1}{2}v_2^i+(\delta_1+\varphi)v_1^i\right] \ , \tag{4.71}$$

$$T^0_{\ i} = \bar{\rho}\left[v_{1i}+\frac{1}{2}v_{2i}+(\delta_1-3\varphi)v_{1i}\right] \ , \tag{4.72}$$

$$T^i_{\ j} = \bar{\rho}\left[v_1^i v_{1j}\right] \ . \tag{4.73}$$

**Finding the scalar constraint on $\Psi_2 - \Phi_2$**

Armed with the first order solutions for $\{\delta_1, v_1\}$ in terms of $\varphi$, we can write down the following second order field equations

$$\Psi_2'' + 3\mathcal{H}\Psi_2' + a^2\Lambda\Psi_2 = \frac{8\pi a^2\bar{\rho}}{3}v_1^2 + \mathcal{H}(\Psi_2' - \Phi_2') + \frac{1}{3}\nabla^2(\Psi_2 - \Phi_2) + (\varphi')^2 + 8\mathcal{H}\varphi\varphi'$$
$$+ a^2\Lambda(\Psi_2 - \Phi_2) + \frac{7}{3}(\nabla\varphi)^2 + \frac{8}{3}\varphi\nabla^2\varphi + 4a^2\Lambda\varphi^2 \,, \tag{4.74}$$

and

$$\frac{1}{2}\partial^i\partial_j(\Psi_2 - \Phi_2) + 2\partial^i\varphi\partial_j\varphi + 4\varphi\partial^i\partial_j\varphi - \frac{1}{3}\delta^i{}_j\left[\frac{1}{2}\nabla^2(\Psi_2 - \Phi_2) + 2(\nabla\varphi)^2 + 4\varphi\nabla^2\varphi\right]$$
$$= 8\pi a^2\bar{\rho}\left(v_1^i v_{1j} - \frac{1}{3}\delta^i{}_j v_1^2\right) . \tag{4.75}$$

The procedure for solving these equations is to find an expression for the combination $\Psi_2 - \Phi_2$ in terms of $\varphi$ by applying the operator $\partial_i\partial^j$ to Equation (4.75). Following [89], it is convenient to define the following quantities,

$$P^i_j = 2\partial^i\varphi\partial_j\varphi + 8\pi a^2\bar{\rho}v_1^i v_{1j} \,, \tag{4.76}$$
$$P = P^i_i \tag{4.77}$$
$$\nabla^2 N = \partial_i\partial^j P^i_j \,, \tag{4.78}$$
$$\nabla^2 Q = -P + 3N \,. \tag{4.79}$$

It is then easy to show that

$$\Psi_2 - \Phi_2 = -4\varphi^2 + Q \,. \tag{4.80}$$

Given this relation, we can translate the combination $\Psi_2 - \Phi_2$ into quadratic products of first order solutions, yielding an evolution equation that is formally the same as the first order equation, just with inhomogeneous source terms. After some calculation, we find

$$\mathcal{H}(\Psi_2' - \Phi_2)' = -\,8\mathcal{H}\varphi\varphi' + \mathcal{H}Q' \,, \tag{4.81}$$
$$\frac{1}{3}\nabla^2(\Psi_2 - \Phi_2) = -\,\frac{8}{3}(\nabla\varphi)^2 - \frac{8}{3}\varphi\nabla^2\varphi - \frac{1}{3}P + N \,. \tag{4.82}$$

These relations and the first order velocity solution in turn allow us to recast the evolution equation, Equation (4.74), as

$$\Psi_2'' + 3\mathcal{H}\Psi_2' + a^2\Lambda\Psi_2 = \mathcal{H}Q' + a^2Q\Lambda + N + (\varphi')^2 - (\nabla\varphi)^2 \,. \tag{4.83}$$

All that remains is to calculate $Q$, $P$ and $N$ in terms of $\varphi_0$. Further algebraic manipulations reveal

$$\begin{aligned}
\mathcal{H}Q' + a^2\Lambda Q = 2g^2\mathcal{H}^2\Omega_m\Big(2\frac{(f-1)^2}{\Omega_m} - \frac{3}{\Omega_m} + 3\Big) \\
\times\Big(\nabla^{-2}\partial^i\varphi_0\partial_i\varphi_0 - 3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\Big)\,, \qquad (4.84)\\
N = \frac{4}{3}g^2\Big(\frac{f^2}{\Omega_m} + \frac{3}{2}\Big)\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\,, \qquad (4.85)\\
P = \frac{4}{3}g^2\Big(\frac{f^2}{\Omega_m} + \frac{3}{2}\Big)\partial^i\varphi_0\partial_i\varphi_0\,. \qquad (4.86)
\end{aligned}$$

Defining the useful quantity $\boldsymbol{\Theta}_0$, known as the *GR kernel*, which is defined by

$$\boldsymbol{\Theta}_0 = \frac{1}{2}\Big(\frac{1}{3}\nabla^{-2}\partial^i\varphi_0\partial_i\varphi_0 - \nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\Big)\,, \tag{4.87}$$

we can state the full second order scalar constraint as

$$\Psi_2 - \Phi_2 = -4g^2\varphi_0^2 - 8g^2\Big(\frac{f^2}{\Omega_m} + \frac{3}{2}\Big)\boldsymbol{\Theta}_0\,, \tag{4.88}$$

and the evolution as

$$\Psi_2'' + 3\mathcal{H}\Psi_2' + a^2\Lambda\Psi_2 = \mathcal{S}(\boldsymbol{x},\tau)\,, \tag{4.89}$$

where the source function $\mathcal{S}(\boldsymbol{x},\tau)$ is defined by

$$\begin{aligned}
\mathcal{S} =& g^2\Omega_m\mathcal{H}^2\Big(\frac{(f-1)^2}{\Omega_m}\varphi_0 + 12\Big(2\frac{(f-1)^2}{\Omega_m} - \frac{3}{\Omega_m} + 3\Big)\boldsymbol{\Theta}_0\Big) \\
&+ g^2\Big(\frac{4}{3}\Big(\frac{f^2}{\Omega_m} + \frac{3}{2}\Big)\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\Big) - g^2(\nabla\varphi_0)^2\,. \qquad (4.90)
\end{aligned}$$

#### Solution method

Equation (4.89) can be solved simply via the method of partial solutions. This procedure is given in detail in [88]. First, we take note of the fact that the homogenous equation (the case where $\mathcal{S} = 0$) is identical to the first order case, therefore our task is simply to find a particular solution for the known source term given in Equation

(4.90). We then take note of the fact that there are four terms, each with different time dependencies, and that these terms are all separable into time-dependent and spatially dependent parts. We are therefore motivated to make an ansatz of the form

$$\Psi_2 = \sum_{n=1}^{4} \mathcal{B}_n(\tau)\mathcal{S}_n(\boldsymbol{x}) + \Psi_{2\text{in}}\frac{g}{g_{\text{in}}} \,, \tag{4.91}$$

where

$$\mathcal{S}_1 = \varphi_0^2 \,, \tag{4.92}$$

$$\mathcal{S}_2 = 6\,\boldsymbol{\Theta}_0 \,, \tag{4.93}$$

$$\mathcal{S}_3 = \nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0) \,, \tag{4.94}$$

$$\mathcal{S}_4 = \partial_i\varphi_0\partial^i\varphi_0 \,. \tag{4.95}$$

Here the $\mathcal{B}(\tau)$ are undetermined functions of conformal time only. Since $\frac{g}{g_{\text{in}}}$ solves the homogeneous equation by definition, we are free to add on any spatial dependency we choose multiplied by this term, and Equation (4.89) will still be solved. It can therefore be seen that if the $\mathcal{B}_n$ satisfy:

$$\mathcal{B}_1'' + 3\mathcal{H}\mathcal{B}_1' + a^2\Lambda\mathcal{B}_1 = g^2\mathcal{H}^2(f-1)^2 \,, \tag{4.96}$$

$$\mathcal{B}_2'' + 3\mathcal{H}\mathcal{B}_2' + a^2\Lambda\mathcal{B}_2 = 2g^2\mathcal{H}^2\big(2(f-1)^2 + 3(\Omega_m - 1)\big) \,, \tag{4.97}$$

$$\mathcal{B}_3'' + 3\mathcal{H}\mathcal{B}_3' + a^2\Lambda\mathcal{B}_3 = \frac{4}{3}\Big(\frac{f^2}{\Omega_m} + \frac{3}{2}\Big) \,, \tag{4.98}$$

$$\mathcal{B}_4'' + 3\mathcal{H}\mathcal{B}_4' + a^2\Lambda\mathcal{B}_4 = -\,g^2 \,, \tag{4.99}$$

then the sum in (4.91) will be a particular solution of the second order evolution equation, (4.89). In this way, the problem is reduced to finding four separate particular solutions of Equation (4.89), each of which is referred to as a *partial solution.*

It is convenient to make the variable transformation $b_n = a\mathcal{B}_n$. The $b_i(\tau)$ can then be shown to satisfy

$$b_1'' + \mathcal{H}b_1' - \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}b_1 = \frac{\mathcal{D}^2\mathcal{H}^2}{a}(f-1)^2 \ , \tag{4.100}$$

$$b_2'' + \mathcal{H}b_2' - \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}b_2 = 2\frac{\mathcal{D}^2\mathcal{H}^2}{a}\big(2(f-1)^2 + 3(\Omega_m - 1)\big) \ , \tag{4.101}$$

$$b_3'' + \mathcal{H}b_3' - \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}b_3 = 2\frac{\mathcal{D}^2}{a} + \frac{4\mathcal{D}'^2}{3\mathcal{H}_0^2\Omega_{m0}} \ , \tag{4.102}$$

$$b_4'' + \mathcal{H}b_4' - \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}b_4 = -\frac{\mathcal{D}^2}{a} \ . \tag{4.103}$$

Starting with $b_4$, we can define $\mathcal{F} = -\frac{3}{2}\mathcal{H}_0^2\Omega_{m0}b_4$, in which case, Equation (4.103) implies

$$\mathcal{F}'' + \mathcal{H}\mathcal{F}' - \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}\mathcal{F} = \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}\mathcal{D}^2 \ , \tag{4.104}$$

which is the well known equation for the second order displacement field in the Newtonian treatment of second order Lagrangian perturbation theory. The solution is calculated via an ingenious method; it turns out the problem can be related to a Taylor series expansion of the solution of the exactly solvable spherical collapse problem (or the equivalent Tolman-Bondi equation). The procedure is carried out in [93] for both Einstein-de Sitter and ΛCDM backgrounds, and the result can be given as follows:

$$\mathcal{F} = \mathcal{D}^2\left[\frac{\Omega_m}{4} - \frac{\Omega_\Lambda}{2} - \frac{1}{U_{3/2}}\left[1 - \frac{3}{2}\frac{U_{5/2}}{U_{3/2}}\right]\right] \ , \tag{4.105}$$

where the function $U_\alpha(\Omega_m, \Lambda)$ is defined as the closed form integral

$$U_\alpha(\Omega_m, \Omega_\Lambda) = \int_0^1 \mathrm{d}x \left[\frac{\Omega_m}{x} + \Omega_\Lambda x + 1 - \Omega_m - \Omega_\Lambda\right]^{-\alpha} \ . \tag{4.106}$$

In the flat ΛCDM case we are interested in, it turns out that $U_\alpha(\Omega_m, 1-\Omega_m)$ also has a representation in terms of the hypergeometric function,

$$U_\alpha(\Omega_m, 1-\Omega_m) = \left(\frac{1}{1+\alpha}\right){}_2F_1\left(1, \alpha, \frac{\alpha+4}{3}, 1-\Omega_m\right) \ . \tag{4.107}$$

Having calculated $b_4$, we can then turn our attention to $b_3$. Using (4.104), background equations, and the ODE satisfied by the first order growth factor $\mathcal{D}$, it can

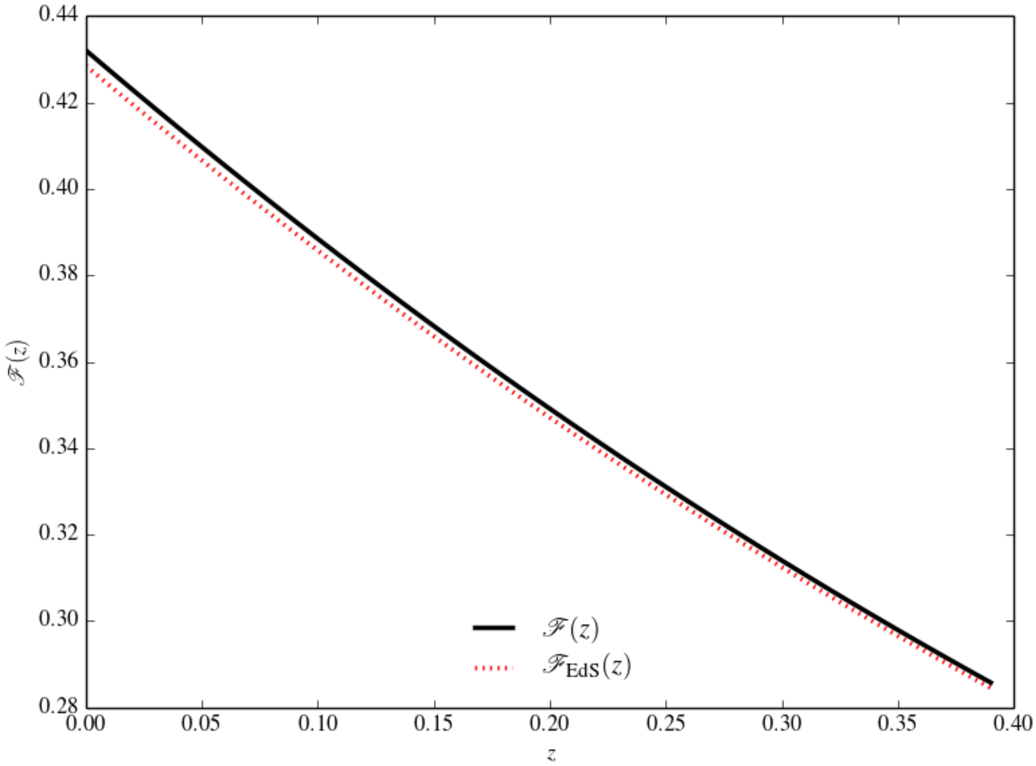


Figure 4.3.: A plot of the ΛCDM displacement, $\mathcal{F}(z)$ against the Einstein-de Sitter version of the same quantity. We can see by eye that broadly speaking, this function is not particularly sensitive to the presence of the cosmological constant.

be shown that the ansatz

$$b_3 = \frac{2}{3\mathcal{H}_0^2\Omega_{m0}}(\mathcal{F} + \mathcal{D}^2) , \tag{4.108}$$

solves (4.102). Similarly, we can show that

$$b_2 = -2\mathcal{D}(g_{\rm in} - g) , \tag{4.109}$$

and

$$b_1 = -\mathcal{D}g + \frac{1}{3}\mathcal{D}g_{\rm in} + \frac{2}{3\mathcal{H}_0^2\Omega_{m0}}\mathcal{D}'^2 , \tag{4.110}$$

where the required factors of $g_{\rm in}$ are fixed by the requirement that $b_1 = b_2 = 0$ in the Einstein-de Sitter limit.

**Initial conditions at second order**

Having obtained the four partial solutions required by the ansatz we made in Equation (4.91), the only remaining problem is the choice of the second order initial condition $\Psi_{2\mathrm{in}}$.

When solving for the evolution, more care must be taken with regards to the initial conditions compared to Newtonian perturbation theory where we were free to choose arbitrary spatial initial conditions. This is due to the existence of additional constraint equations in general relativity which must also be satisfied compared to Newtonian perturbation theory. An excellent description of the procedure for fixing initial conditions is given in [94], which we will summarise here.

Initial conditions are traditionally fixed at a time when the cosmological perturbations relevant for large scale structure in the universe today were well outside the horizon [95]. The statistical characteristics of the seeds of these density fluctuations are predicted by various models of cosmic inflation. In order to connect models of inflation with the initial conditions for structure formation, it is convenient to use the gauge invariant *curvature perturbation of uniform density hypersurfaces*, $\zeta = \zeta_1 + \frac{1}{2}\zeta_2 + \dots$, where $\zeta_1 = -\Psi_1 - \mathcal{H}\frac{\delta\rho_1}{\bar{\rho}'}$ and $\zeta_2$ is defined by

$$\zeta_2 = -\Psi_2 - \mathcal{H}\frac{\delta\rho_2}{\bar{\rho}'} + 2\mathcal{H}\frac{\delta\rho_1'}{\bar{\rho}'}\frac{\delta\rho_1}{\bar{\rho}'} + 2\frac{\delta\rho_1}{\bar{\rho}'}(\Psi_1' + 2\mathcal{H}\Psi_1) - \left(\frac{\delta\rho_1}{\bar{\rho}'}\right)^2\left(\mathcal{H}\frac{\bar{\rho}''}{\bar{\rho}} - \mathcal{H}' - 2\mathcal{H}^2\right), \tag{4.111}$$

where our definition coincides with the one given in the paper [89]. Critically, the gauge invariant curvature perturbation remains constant on super-horizon scales after it has been generated by inflation (assuming isocurvature perturbations are not present[2]). This allows us to set initial conditions as soon as $\zeta$ becomes constant. A common way to distinguish between different models of inflation is to look at the amount of *primordial nongaussianity* they generate. We will come to understand the effects of primordial nongaussianity on the statistical correlation functions that are related to cosmological observables in the coming section on statistics, but for now, we content ourselves with a parameterised description. For more information about primordial nongaussianity, see the review article [97].

The second order gauge invariant curvature perturbation $\zeta_2$ encodes information about the primordial non-gaussianity generated by inflation. Standard single field inflation results in $\zeta_2 \approx 2\zeta_1^2$. The normal way of parameterising the amount of

[2]For a discussion of the effects of isocurvature perturbations, see [96]

primordial non-gaussianity is by setting

$$\zeta_2 = 2a_{\rm nl}\zeta_1^2 \ , \tag{4.112}$$

whereby the results of different inflationary models are encoded in the parameter $a_{\rm nl}$ [89] [88]. The parameter $a_{\rm nl}$ can be connected to a different nongaussianity parameter, $f_{\rm nl}$, by the following relation,

$$f_{\rm nl} = \frac{5}{3}(a_{\rm nl} - 1) \ . \tag{4.113}$$

Different inflationary scenarios will then give rise to different numerical values of $f_{\rm nl}$ and consequently $a_{\rm nl}$. Cosmologists hope to ultimately constrain these arbitrary parameters by comparing the observed large scale structure to the predictions given by different initial conditions.

On large scales, during the matter-dominated era,

$$\zeta_1 = -\frac{5}{3}\varphi_{\rm in} = -\frac{5}{3}g_{\rm in}\varphi_0 \ , \tag{4.114}$$

where $\varphi_{\rm in} = g_{\rm in}\varphi_0$ is the numerically constant value of the gravitational potential deep in the matter dominated phase of the universe's evolution, where we choose to select our initial conditions. A good approximation for $g_{\rm in}$ is given in [89] as

$$g_{\rm in} \sim \frac{2}{5}\Omega_{m0}^{-1}\big(\Omega_{m0}^{4/7} + \frac{3}{2}\Omega_{m0}\big) \ . \tag{4.115}$$

We can therefore write

$$\zeta_2 = \frac{50}{9}a_{\rm nl}g_{\rm in}^2\varphi_0^2 \ . \tag{4.116}$$

Given this specific value for $\zeta_2$, we can use (4.111) deep during the matter-dominated era, in conjunction with the following Einstein de-Sitter limits of the constraint and 00 field equations

$$\Psi_{2\rm in} - \Phi_{2\rm in} = -4g_{\rm in}^2\varphi_0^2 - 20\, g_{\rm in}^2\mathbf{\Theta}_0 \ , \tag{4.117}$$

$$\frac{1}{3}\nabla^2\Psi_{2\rm in} - \mathcal{H}^2\Phi_{2\rm in} = \frac{4\pi a^2\bar{\rho}}{3}\delta_2 + \frac{8\pi a^2\bar{\rho}}{3}v_{1i}v_1^i - 4\mathcal{H}^2\varphi^2 - \frac{8}{3}\varphi\nabla^2\varphi - \nabla\varphi \ , \tag{4.118}$$

to eliminate $\delta_2$, resulting in the following expression for $\Phi_{2\mathrm{in}}$:

$$\Phi_{2\mathrm{in}} = -\frac{3}{5}\zeta_2 + \frac{16}{3} g_{\mathrm{in}}^2 \varphi_0^2 + 12 g_{\mathrm{in}}^2 \boldsymbol{\Theta}_0 \ . \tag{4.119}$$

Plugging in our parameterised value for $\zeta_2$, we obtain

$$\Phi_{2\mathrm{in}} = 2 g_{\mathrm{in}}^2 \varphi_0^2 \Big( -\frac{5}{3}(a_{\mathrm{nl}} - 1) - 1 \Big) + 12 g_{\mathrm{in}}^2 \, \boldsymbol{\Theta}_0^2 \ , \tag{4.120}$$

and consequently

$$\Psi_{2\mathrm{in}} = 2 g_{\mathrm{in}}^2 \varphi_0^2 \Big( -\frac{5}{3}(a_{\mathrm{nl}} - 1) - 1 \Big) - 8 g_{\mathrm{in}}^2 \, \boldsymbol{\Theta}_0^2 \ . \tag{4.121}$$

These expressions demonstrably therefore satisfy both the constraints from the 00-field equation and Equation (4.117), and are therefore valid choices of initial condition.

### Full second-order solutions

At this point we will define the so-called *Newtonian kernel*

$$\boldsymbol{\Psi}_0 = -\frac{1}{2}\nabla^{-2}\Big[ (\nabla^2\varphi_0)^2 - \partial_i\partial_j\varphi_0\partial^i\partial^j\varphi_0 \Big] \ , \tag{4.122}$$

satisfying the identities

$$\nabla^{-2}\Big( \partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0) \Big) = -2\boldsymbol{\Psi}_0 + (\nabla\varphi_0)^2 \ , \tag{4.123}$$

$$\nabla^2\boldsymbol{\Theta}_0 = \boldsymbol{\Psi}_0 - \frac{1}{3}\partial_i\varphi_0\partial^i\varphi_0 \ . \tag{4.124}$$

Using these identities we can finally write the full second order solution for the gravitational potentials as

$$\begin{aligned}
\Psi_2 =& \Big( -g^2 + \frac{5}{3} g g_{\mathrm{in}}(1 - 2_{\mathrm{nl}}) + \frac{2\,\mathcal{D}'^2}{3a\mathcal{H}_0^2\Omega_{m0}} \Big)\varphi_0^2 + 6\Big( 2g^2 - \frac{10}{3} g g_{\mathrm{in}} \Big)\boldsymbol{\Theta}_0 \\
&+ \frac{2\mathcal{D}^2}{3a\mathcal{H}_0^2\Omega_{m0}}(\nabla\varphi_0)^2 - \frac{4(\mathcal{D}^2 + \mathcal{F})}{3a\mathcal{H}_0^2\Omega_{m0}}\boldsymbol{\Psi}_0 \ ,
\end{aligned} \tag{4.125}$$

$$\begin{aligned}
\Phi_2 =& \Big( 3g^2 + \frac{5}{3} g g_{\mathrm{in}}(1 - 2_{\mathrm{nl}}) + \frac{2\,\mathcal{D}'^2}{3a\mathcal{H}_0^2\Omega_{m0}} \Big)\varphi_0^2 + 8g^2\Big( \frac{f^2}{\Omega_{m0}} + \frac{3}{2} \Big)\boldsymbol{\Theta}_0 \\
&+ \frac{2\mathcal{D}^2}{3a\mathcal{H}_0^2\Omega_{m0}}(\nabla\varphi_0)^2 - \frac{4(\mathcal{D}^2 + \mathcal{F})}{3a\mathcal{H}_0^2\Omega_{m0}}\boldsymbol{\Psi}_0 \ .
\end{aligned} \tag{4.126}$$

We can now use the constraint from the 00 Einstein field equation to calculate the solution for $\delta_2$. For the purposes of calculation, we will define

$$\Psi_2 = A_1\varphi_0^2 + A_2\mathbf{\Theta}_0 + A_3(\nabla\varphi_0)^2 + A_4\mathbf{\Psi}_0 \,, \tag{4.127}$$

so that

$$\Phi_2 = (A_1 + 4g^2)\varphi_0^2 + \Big(A_2 + \frac{8\mathcal{D}'^2}{a\mathcal{H}_0^2\Omega_{m0}} + 12g^2\Big)\mathbf{\Theta}_0 + A_3(\nabla\varphi_0)^2 + A_4\mathbf{\Psi}_0 \,, \tag{4.128}$$

and

$$A_1 = -g^2 + \frac{5}{3}gg_{\rm in}(1-2_{\rm nl}) + \frac{2\,\mathcal{D}'^2}{3a\mathcal{H}_0^2\Omega_{m0}} \,, \tag{4.129}$$

$$A_2 = 6\Big(2g^2 - \frac{10}{3}gg_{\rm in}\Big) \,, \tag{4.130}$$

$$A_3 = \frac{2\mathcal{D}^2}{3a\mathcal{H}_0^2\Omega_{m0}} \,, \tag{4.131}$$

$$A_4 = -\frac{4(\mathcal{D}^2 + \mathcal{F})}{3a\mathcal{H}_0^2\Omega_{m0}} \,. \tag{4.132}$$

The required constraint field equation is

$$\begin{aligned}\frac{1}{3}\Psi_2 - \mathcal{H}\Psi_2' - \mathcal{H}^2\Phi_2 = \frac{4\pi a^2\bar{\rho}}{3}\delta_2 + \frac{8\pi a^2\bar{\rho}}{3}v_1^2 - 4\mathcal{H}^2\varphi^2 \\ - (\varphi')^2 - \frac{8}{3}\varphi\nabla^2\varphi - (\nabla\varphi)^2 \,.\end{aligned} \tag{4.133}$$

Inserting our solutions for $\Psi_2$, and $\Phi_2$, along with the first order expression for $v_{1i}$, and using the identity (4.124), we can write the expression

$$\begin{aligned}\delta_2 = &\frac{2a}{\mathcal{H}_0^2\Omega_{m0}}\Big[-\mathcal{H}A_1' - \mathcal{H}^2A_1 + (g')^2\Big]\varphi_0^2 \\ &+ \frac{2a}{\mathcal{H}_0^2\Omega_{m0}}\Big[-\mathcal{H}A_2' - \mathcal{H}^2A_2 - \frac{8\mathcal{H}^2(\mathcal{D}')^2}{a\mathcal{H}_0^2\Omega_{m0}} - 12\mathcal{H}^2g^2\Big]\mathbf{\Theta}_0 \\ &+ \frac{2a}{\mathcal{H}_0^2\Omega_{m0}}\Big[\frac{2}{3}A_1 - \frac{1}{9}A_2 - \mathcal{H}A_3' - \mathcal{H}^2A_3 - \frac{4(\mathcal{D}')^2}{9a\mathcal{H}_0^2\Omega_{m0}} + g^2\Big](\nabla\varphi_0)^2 \\ &+ \frac{2a}{\mathcal{H}_0^2\Omega_{m0}}\Big[\frac{1}{3}A_2 - \mathcal{H}A_4' - \mathcal{H}^2A_4\Big]\mathbf{\Psi}_0 + \frac{2a}{\mathcal{H}_0^2\Omega_{m0}}\Big[\frac{2}{3}A_1 + \frac{8}{3}g^2\Big]\varphi_0\nabla^2\varphi_0 \\ &+ \frac{2a}{\mathcal{H}_0^2\Omega_{m0}}\Big[-\frac{A_4}{6}(\nabla^2\varphi_0)^2 + \frac{2A_3}{3}\partial_i\varphi_0\nabla^2\partial\varphi_0 + \Big(\frac{2}{3}A_3 + \frac{A_4}{6}\Big)\partial_i\partial_j\varphi_0\partial^i\partial^j\varphi_0\Big] \,.\end{aligned} \tag{4.134}$$

This expression will be useful for comparison with calculations performed in later chapters. Substituting in the definitions of the $A_i$, using identity (4.62) to replace factors of $g_{\rm in}$ and calculating the appropriate derivatives, we finally find the real space expression for the second order density contrast in Poisson gauge to be given by

$$\begin{aligned}\delta_2 = &\frac{2\mathcal{H}^2\mathcal{D}^2}{a\mathcal{H}_0^2\Omega_{\rm m0}}\left[f^2-4f\right]\varphi_0^2 + \frac{10\mathcal{H}}{3\mathcal{H}_0^2\Omega_{\rm m0}}(1+2a_{\rm nl})\mathcal{D}' g_{\rm in}\varphi_0^2 - \frac{24\mathcal{H}\mathcal{D}'\mathcal{D}}{a\mathcal{H}_0^2\Omega_{\rm m0}}\boldsymbol{\Theta}_0\\ &+\frac{2\mathcal{D}}{\mathcal{H}_0^2\Omega_{\rm m0}}\left[g-\frac{20}{9}g_{\rm in}a_{\rm nl}\right](\nabla\varphi_0)^2 + \frac{8\mathcal{H}\mathcal{F}'}{3(\mathcal{H}_0^2\Omega_{\rm m0})^2}\boldsymbol{\Psi}_0\\ &+\frac{2}{3\mathcal{H}_0^2\Omega_{\rm m0}}\left[6\mathcal{D}g+\frac{10}{3}\mathcal{D}g_{\rm in}(1-2a_{\rm nl})+\frac{4}{3}\frac{(\mathcal{D}')^2}{\mathcal{H}_0^2\Omega_{\rm m0}}\right]\varphi_0\nabla^2\varphi_0\\ &+\frac{4}{9(\mathcal{H}_0^2\Omega_{\rm m0})^2}\left[(\mathcal{D}^2+\mathcal{F})(\nabla^2\varphi_0)^2+2\mathcal{D}^2\partial_i\varphi_0\nabla^2\partial^i\varphi_0+(\mathcal{D}^2-\mathcal{F})\partial_i\partial_j\varphi_0\partial^i\partial^j\varphi_0\right].\end{aligned} \tag{4.135}$$

One can use the second order $0i$ Einstein field equation to perform a similar exercise here, however we will not do so, instead focussing on understanding properties of the second-order density contrast.

### Fourier-space kernel for the second-order density contrast

For statistical purposes it is useful to express this solution in the same form that we expressed the solution to the second order Newtonian density contrast, namely via a Fourier-space integral kernel. This procedure is performed in [98], so we refer the reader to that paper for reference (although we have used a slightly different notational system and convention regarding the placement of a factor of two). We will again adopt the strategy of compactifying our expression via defining

$$\begin{aligned}\delta_2 = &J_1\,\varphi_0^2 + J_2\,\boldsymbol{\Theta}_0 + J_3\,(\nabla\varphi_0)^2 + J_4\,\boldsymbol{\Psi}_0 + J_5\,\varphi_0\nabla^2\varphi_0\\ &+J_6\,(\nabla^2\varphi_0)^2 + J_7\,\partial_i\varphi_0\nabla^2\partial^i\varphi_0 + J_8\,\partial_i\partial_j\varphi_0\partial^i\partial^j\varphi_0\,,\end{aligned} \tag{4.136}$$

where the $J_n$ are the following functions of conformal time only:

$$J_1 = \frac{2\mathcal{H}^2\mathcal{D}^2}{a\mathcal{H}_0^2\Omega_{\rm m0}}\left[f^2-4f\right] + \frac{10\mathcal{H}}{3\mathcal{H}_0^2\Omega_{\rm m0}}(1+2a_{\rm nl})\mathcal{D}'g_{\rm in}\ , \tag{4.137}$$

$$J_2 = -\frac{24\mathcal{H}\mathcal{D}'\mathcal{D}}{a\mathcal{H}_0^2\Omega_{\rm m0}}\ , \tag{4.138}$$

$$J_3 = \frac{2\mathcal{D}}{\mathcal{H}_0^2\Omega_{\rm m0}}\left[g - \frac{20}{9}g_{\rm in}a_{\rm nl}\right]\ , \tag{4.139}$$

$$J_4 = \frac{8\mathcal{H}\mathcal{F}'}{3(\mathcal{H}_0^2\Omega_{\rm m0})^2}\ , \tag{4.140}$$

$$J_5 = \frac{2}{3\mathcal{H}_0^2\Omega_{\rm m0}}\left[6\mathcal{D}g + \frac{10}{3}\mathcal{D}g_{\rm in}(1-2a_{\rm nl}) + \frac{4}{3}\frac{(\mathcal{D}')^2}{\mathcal{H}_0^2\Omega_{\rm m0}}\right]\ , \tag{4.141}$$

$$J_6 = \frac{4}{9(\mathcal{H}_0^2\Omega_{\rm m0})^2}(\mathcal{D}^2+\mathcal{F})\ , \tag{4.142}$$

$$J_7 = \frac{8}{9(\mathcal{H}_0^2\Omega_{\rm m0})^2}\mathcal{D}^2\ , \tag{4.143}$$

$$J_8 = \frac{4}{9(\mathcal{H}_0^2\Omega_{\rm m0})^2}(\mathcal{D}^2-\mathcal{F})\ . \tag{4.144}$$

Substituting in the definitions of $\mathbf{\Theta}_0$ and $\mathbf{\Psi}_0$, and taking the Fourier transform, we obtain

$$\begin{aligned}\delta_2(k) = &\int \mathrm{d}^3q_1\,\mathrm{d}^3q_2\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\,\varphi_0(\boldsymbol{q}_1)\,\varphi_0(\boldsymbol{q}_2)\times \\ &\Bigg[J_1 + \frac{(k^2\,J_4 - J_2)}{2k^4}\,q_1^2q_2^2\left(1-(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\right) - \frac{(J_2+k^2\,J_3)\,q_1q_2}{3k^2}(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2) \\ &- \frac{J_5}{2}(q_1^2+q_2^2) + J_6\,q_1^2q_2^2 + J_7\,(q_1^2+q_2^2)q_1q_2(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2) + J_8\,q_1^2q_2^2(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\Bigg]\ .\end{aligned} \tag{4.145}$$

The product $\varphi_0(\boldsymbol{q}_1)\varphi_0(\boldsymbol{q}_2)$ can be evaluated using the first order generalisation of the Poisson equation

$$\varphi_0(\boldsymbol{q}_1)\varphi_0(\boldsymbol{q}_2) = \frac{9}{4}\frac{(\mathcal{H}_0^2\Omega_{m0})^2}{\mathcal{D}^2q_1^2q_2^2}\left(1+\frac{3\mathcal{H}^2f}{q_1^2}\right)^{-1}\left(1+\frac{3\mathcal{H}^2f}{q_2^2}\right)^{-1}\delta_1(\boldsymbol{q}_1)\delta_1(\boldsymbol{q}_2)\ . \tag{4.146}$$

Using Equation (4.146) and the following triangular vector identities,

$$\frac{1}{q_1 q_2} = \frac{1}{k^2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)+\frac{2}{k^2}\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2\ , \tag{4.147}$$
$$\frac{1}{q_1^2 q_2^2} = \frac{1}{k^4}\left(4+\left(\frac{q_1}{q_2}-\frac{q_2}{q_1}\right)^2+4\,\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)+4(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\right)\ ,$$

we find that we can write $\delta_2(k,\tau)$ in the form

$$\delta_2(k,\tau) = \int \mathrm{d}^3q_1\,\mathrm{d}^3q_2\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\,\delta_1(\boldsymbol{q}_1)\delta_1(\boldsymbol{q}_2)\mathcal{K}_2(\boldsymbol{q}_1,\boldsymbol{q}_2,\tau)\ , \tag{4.148}$$

where $\mathcal{K}_2(\boldsymbol{q}_1,\boldsymbol{q}_2,\tau)$ plays the same role that was played by $F_2(\boldsymbol{q}_1,\boldsymbol{q}_2,)$ in Newtonian perturbation theory, and is given by

$$\mathcal{K}_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{(\beta-\alpha)+\frac{\beta}{2}\,\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)+\alpha\left(\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2\right)^2+\gamma\left(\frac{k_1}{k_2}-\frac{k_2}{k_1}\right)^2}{\left(1+\frac{3\mathcal{H}^2 f}{q_1^2}\right)\left(1+\frac{3\mathcal{H}^2 f}{q_2^2}\right)}\ , \tag{4.149}$$

where the functional dependencies of $\alpha(k,\tau)$, $\beta(k,\tau)$ and $\gamma(k,\tau)$ have been suppressed for space-saving purposes - i.e the terms in brackets should be understood to be multiplying these functions. $\alpha(k,\tau)$, $\beta(k,\tau)$ and $\gamma(k,\tau)$ are then given by

$$\alpha(k,\tau) = \left(1-\frac{\mathcal{F}}{\mathcal{D}^2}\right)+\frac{(\mathcal{H}_0^2\Omega_{m0})^2}{\mathcal{D}^2}\left[-\frac{9(4J_3+J_4)}{8k^2}+\frac{9J_1}{k^4}-\frac{3J_2}{8k^4}\right]\ , \tag{4.150}$$

$$\beta(k,\tau) = 2+\frac{(\mathcal{H}_0^2\Omega_{m0})^2}{\mathcal{D}^2}\left[-\frac{9(J_3+J_5)}{2k^2}+\frac{18J_1}{k^4}-\frac{3J_2}{2k^4}\right]\ , \tag{4.151}$$

$$\gamma(k,\tau) = \frac{(\mathcal{H}_0^2\Omega_{m0})^2}{\mathcal{D}^2}\left[-\frac{9J_5}{8k^2}+\frac{9J_1}{4k^4}\right]\ . \tag{4.152}$$

This expression is equivalent to the one calculated in [98] up to a conventional factor of two.

The relativistic correction functions $\alpha(k,\tau)$, $\beta(k,\tau)$ and $\gamma(k,\tau)$ are plotted in Figures 4.4, 4.5 and 4.6, for both ΛCDM background universes and for Einstein-de Sitter. It is evident that relativistic corrections tend to be maximised in the Einstein-de Sitter geometry. Intuitively this makes sense, since the growth of structure is also maximised in that geometry, whilst the effect of accelerated expansion damps the growth of structure.

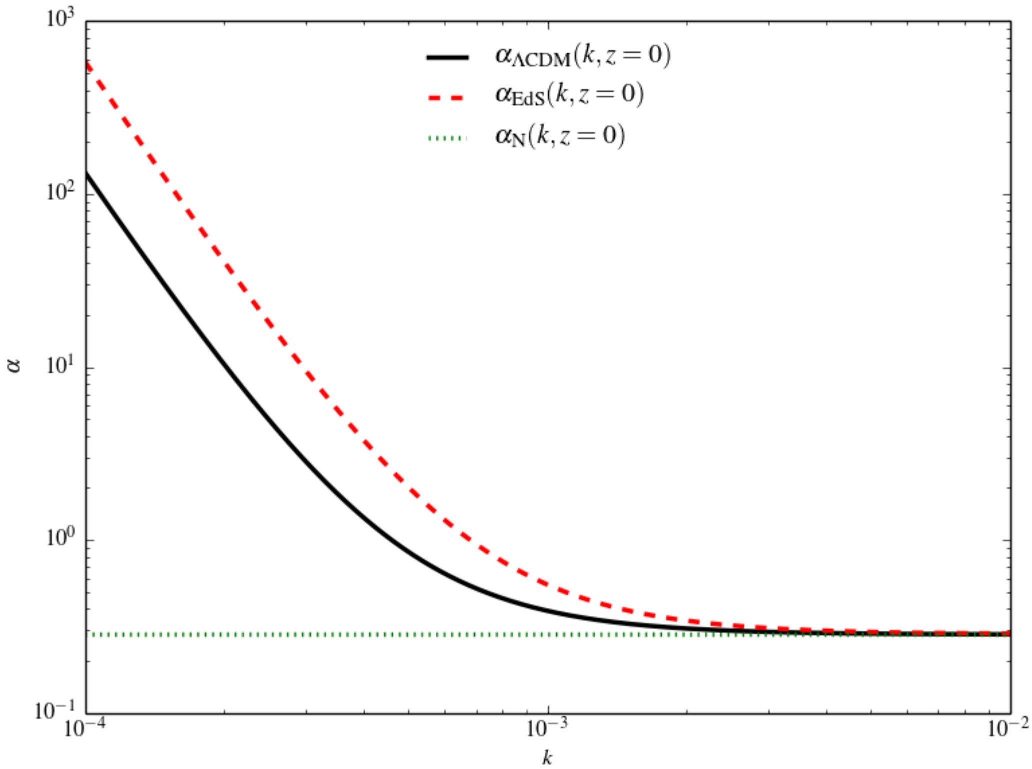


Figure 4.4.: A plot showing the magnitude of the relativistic corrections to $\alpha(z=0)$ in $\Lambda$CDM and Einstein-de Sitter cosmologies. The Newtonian value is $\alpha_{\mathrm{N}} = \frac{4}{7}$ and does not change with scale. We see that the generic effect of the inclusion of the cosmological constant is to reduce the magnitude of relativistic corrections. Relativistic corrections start to become non-negligible when the fraction $\frac{\mathcal{H}^2}{k^2}$ becomes non-negligible.

### Limiting regimes

We will briefly give details of both the Einstein-de Sitter and Newtonian limits of these solutions. Firstly, the Newtonian limit is simply obtained by taking $\frac{\mathcal{H}}{k} \to 0$. This corresponds to consideration of length scales that are extremely small compared to the horizon. In this case, it is easy to see that we recover

$$\mathcal{K}_2^{(\mathrm{N})} = \left(1 + \frac{\mathcal{F}}{\mathcal{D}^2}\right) + \hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2 \left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \left(1 - \frac{\mathcal{F}}{\mathcal{D}^2}\right)(\hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2)^2 \tag{4.153}$$

the generalisation of the Einstein-de Sitter background Newtonian perturbation theory solution to a $\Lambda$CDM background. The Einstein-de Sitter (and no primordial nongaussianity) limit of any expression can be found by making the following sub-

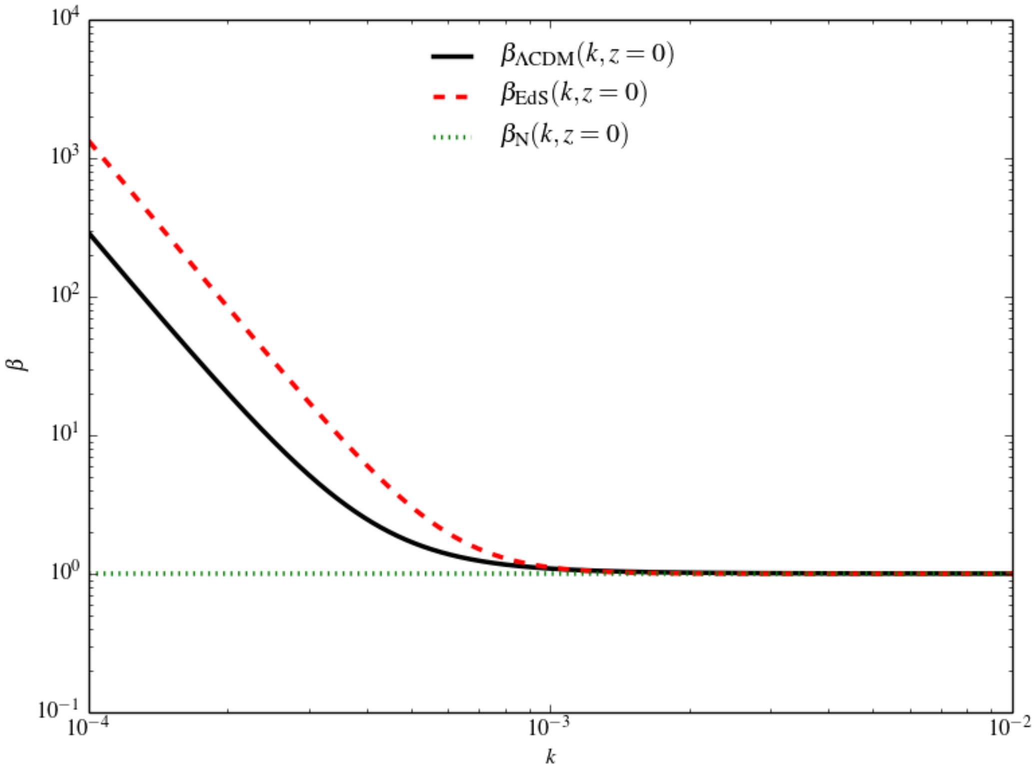


Figure 4.5.: A plot showing the magnitude of the relativistic corrections to $\beta(z=0)$ in ΛCDM and Einstein-de Sitter cosmologies. The Newtonian value is $\beta_{\mathrm{N}} = 2$ and does not change with scale. We see that the generic effect of the inclusion of the cosmological constant is to reduce the magnitude of relativistic corrections. Relativistic corrections start to become non-negligible when the fraction $\frac{\mathcal{H}^2}{k^2}$ becomes non-negligible.

stitutions

$$\mathcal{D} \to a \;, \tag{4.154}$$

$$f \to 1 \;, \tag{4.155}$$

$$g \to 1 \;, \tag{4.156}$$

$$g_{\mathrm{in}} \to 1 \;, \tag{4.157}$$

$$\Omega_{m0} \to 1 \;, \tag{4.158}$$

$$\frac{\mathcal{H}_0^2}{a} \to \mathcal{H}^2 \;, \tag{4.159}$$

$$\mathcal{H}^2 a \to 4 \;, \tag{4.160}$$

$$a_{\mathrm{nl}} \to 1 \;, \tag{4.161}$$

$$\mathcal{F} \to \frac{3}{7}a^2 \;. \tag{4.162}$$

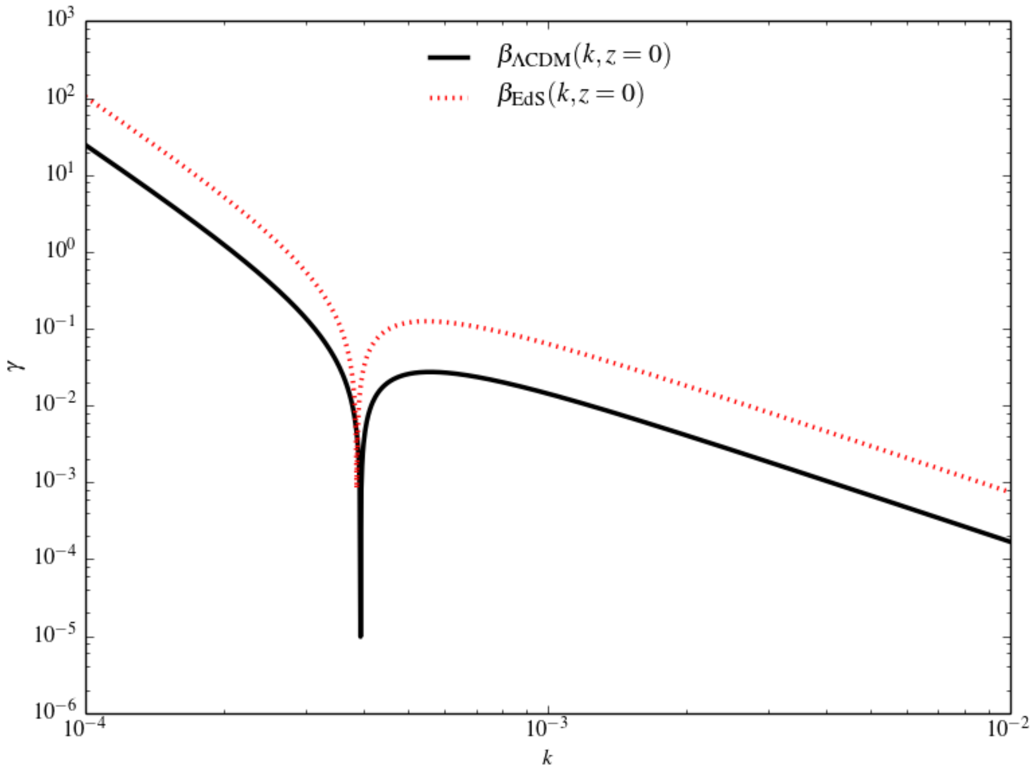


Figure 4.6.: A plot showing the magnitude of the relativistic corrections to $\gamma(z=0)$ in $\Lambda$CDM and Einstein-de Sitter cosmologies. The Newtonian value is $\gamma_{\mathrm{N}} = 0$ (which cannot be plotted on this log scale) and does not change with scale. We see that the generic effect of the inclusion of the cosmological constant is to reduce the magnitude of relativistic corrections. Relativistic corrections start to become non-negligible when the fraction $\frac{\mathcal{H}^2}{k^2}$ becomes non-negligible.

Making these replacements, we find that the Einstein-de Sitter limits of the $b_n$ solutions are as follows:

$$b_{1\mathrm{EdS}} = 0\ , \tag{4.163}$$

$$b_{2\mathrm{EdS}} = 0\ , \tag{4.164}$$

$$b_{3\mathrm{EdS}} = \frac{5}{21}a^2\ , \tag{4.165}$$

$$b_{4\mathrm{EdS}} = -\frac{1}{14}a^2\ . \tag{4.166}$$

From this we conclude that in Einstein-de Sitter universes, the only dynamical evolution of the gravitational potential occurs in the Newtonian part of the solution, and that relativistic effects only occur as a result of the requirement that initial conditions satisfy additional constraint equations. This is not the case in the full $\Lambda$CDM solution, where each partial solution evolves with a different time dependency. We can also evaluate the Einstein-de Sitter limit of the $J_n$'s:

$$J_{1\mathrm{EdS}} = 4 \,, \tag{4.167}$$
$$J_{2\mathrm{EdS}} = -24 \,, \tag{4.168}$$
$$J_{3\mathrm{EdS}} = -\frac{22}{9\mathcal{H}^2} \,, \tag{4.169}$$
$$J_{4\mathrm{EdS}} = \frac{16}{7\mathcal{H}^2} \,. \tag{4.170}$$
$$J_{5\mathrm{EdS}} = \frac{8}{3\mathcal{H}^2} \,, \tag{4.171}$$
$$J_{6\mathrm{EdS}} = \frac{10}{7} \,, \tag{4.172}$$
$$J_{7\mathrm{EdS}} = 2 \,, \tag{4.173}$$
$$J_{8\mathrm{EdS}} = \frac{4}{7} \,. \tag{4.174}$$

We can therefore evaluate the Einstein-de Sitter limit of our expressions for $\alpha$, $\beta$ and $\gamma$ to be

$$\alpha_{\mathrm{EdS}} = \frac{4}{7} + \frac{59}{7}\frac{\mathcal{H}^2}{k^2} + 45\frac{\mathcal{H}^4}{k^4} \,, \tag{4.175}$$
$$\beta_{\mathrm{EdS}} = 2 + 108\frac{\mathcal{H}^2}{k^2} - \frac{\mathcal{H}^4}{k^4} \,, \tag{4.176}$$
$$\gamma_{\mathrm{EdS}} = 9\frac{\mathcal{H}^2}{k^2} - 3\frac{\mathcal{H}^4}{k^4} \,. \tag{4.177}$$

These expressions agree with those calculated in [98] up to a conventional factor of 2.

## 4.3. From solutions to statistics

The kernels calculated in the previous two sections can be used in a variety of contexts [84]. We choose to focus on their application to the calculation of $n$-point correlation of the dark matter density field, a critical part of the process used to theoretically predict observables in galaxy surveys, since galaxies are considered to be a biased tracer of the dark matter. We will demonstrate the basic method for calculating such statistics.

### 4.3.1. Linear matter power spectrum

Let us first start with the *2-point correlation function*, defined by

$$\langle \delta(\boldsymbol{x}_1, \tau)\delta(\boldsymbol{x}_2, \tau)\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\xi_2(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) \ , \tag{4.178}$$

which depends only on the absolute distance between the spatial points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ because of background homogeneity and isotropy. The correlation function $\xi_2$ measures the excess probability of finding two identical values for the overdensity field $\delta$ separated by a distance $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$. Intuitively, assuming that the actual galaxy distribution in the real universe will be related to the underlying dark matter density distribution, one can conceptualise this statistic as being related to the clustering of galaxies. It is often more convenient to work with the Fourier transform of this quantity, the so-called *matter power spectrum*:

$$\langle \delta(\boldsymbol{k}_1, \tau)\delta(\boldsymbol{k}_2, \tau)\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_1 - \boldsymbol{k}_2)P(k, \tau) \ , \tag{4.179}$$

where $P(k, \tau)$ is the scale dependent matter power spectrum. This is because the evolution of each Fourier mode is independent in the linear, Gaussian case. Inserting a generic perturbative expansion, $\delta(k, \tau) = \delta_1(k, \tau) + (1/2)\delta_2(k, \tau) + \ldots$ into this expression, we see that the matter power spectrum can be split into a leading order piece,

$$\langle \delta_1(\boldsymbol{k}_1, \tau)\delta_1(\boldsymbol{k}_2, \tau)\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_1 - \boldsymbol{k}_2)P_L(k, \tau) \ , \tag{4.180}$$

and higher order corrections involving products of higher-order $\delta_n(k, \tau)$, e.g.

$$\langle \delta_2(\boldsymbol{k}_1, \tau)\delta_2(\boldsymbol{k}_2, \tau)\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_1 - \boldsymbol{k}_2)P_{22}(k, \tau) \ . \tag{4.181}$$

Since the perturbative solutions we calculated in the previous section all express $\delta_n(k)$ in terms of products of $\delta_1(k)$, it follows that we can express higher-order statistics in terms of the leading order, *linear* matter power spectrum, $P_L(k)$, and that our choice of $\varphi(k)$ as a gaussian random field is encoded in the statistical distribution of $P_L(k)$.

To formalise this claim, let us introduce the central tool for dealing with gaussian random fields, *Wick's theorem.* Wick's theorem states the following:

- The statistical correlation between a product of odd numbers of gaussian random fields is zero, e.g

$$\langle \delta_1(k_1)\delta_1(k_2)\delta_1(k_3)\rangle = 0 \ . \tag{4.182}$$

- The statistical correlation between a product of even numbers of gaussian fields is given by the expression

$$\langle \delta_1(k_1)\delta_1(k_2)\ldots\delta_1(k_n)\rangle = \sum\prod \text{pairings} \ , \tag{4.183}$$

  where the sum is taken over all possible products of all possible pairings of the gaussian random field in question.

This decomposition is best demonstrated with an example. Consider the reduction of the higher-order correlation function $\langle \varphi_1\varphi_2\varphi_3\varphi_4\rangle$, (where $\varphi_n = \varphi(k_n)$ is some arbitrary gaussian random field):

$$\langle \varphi_1\varphi_2\varphi_3\varphi_4\rangle = \langle \varphi_1\varphi_2\rangle\langle \varphi_3\varphi_4\rangle + \langle \varphi_1\varphi_3\rangle\langle \varphi_2\varphi_4\rangle + \langle \varphi_1\varphi_4\rangle\langle \varphi_2\varphi_3\rangle \ . \tag{4.184}$$

This pattern is easily generalisable to higher order correlation functions. The implication of Wick's theorem is that any purely gaussian random field is entirely characterised by its two-point correlation function. Applied to cosmology, it allows us to calculate higher-order statistics in terms of products of the initial gaussian random field's power spectrum, $P_L(k)$ by reducing higher-order correlation functions to products of this fundamental quantity. Accordingly, to fix our initial conditions, the only fundamental quantity we require is $P_L(k)$, which fortunately is exactly the quantity predicted by various theories of inflation.

Since there are many competing theories of inflation, all leading to slightly different predictions for $P_L(k)$ at the end of inflation, late universe cosmologists often choose to parameterise the form of $P_L(k)$. The chosen form is the *nearly scale invariant power spectrum*:

$$\Delta^2(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1} \ , \tag{4.185}$$

where $\Delta^2(k) = \frac{k^3P(k)}{2\pi^2}$ is the dimensionless form of the power spectrum. Here, $A_s$ is the primordial amplitude of perturbations when they enter the horizon after inflation, and $n_s$ is the *spectral index*, characterising deviation from scale invariance (a perfectly scale invariant power spectrum would by definition have $n_s = 1$). The symbol $k_*$ is known as the *pivot scale* and it here it is set to $k_* = 0.05\,\mathrm{Mpc}^{-1}$ (the default choice for the CosmoMC package [99]). The choice is not arbitrary unless the full posterior distribution over all cosmological parameters is known - since marginalisation occurs during data analysis, the choice of scale matters and should be chosen in such a way as to optimise observational constraints on inflationary parameters [100]. Since our study is purely theoretical, we will continue using $k_* = 0.05\,\mathrm{Mpc}^{-1}$.

If the universe had evolved precisely as $\Lambda$CDM (or Einstein-de Sitter) since the end of inflation, one would be perfectly entitled to use the perturbative solutions derived in the previous sections, together with the choice of the nearly scale-invariant power spectrum initial condition at the conformal time at the end of inflation, $\tau_i$. Unfortunately, the real universe is more complicated, consisting of many more species than cold dark matter and the cosmological constant, and having undergone a significant period of radiation-dominated expansion during which the dynamics of many of these species is non-trivial. In order to model these dynamics accurately, one solves the fully coupled first-order Einstein-Boltzmann system for the full mixture. There are many codes available which numerically integrate this system, including *CLASS* and *SONG*.

Our intention is to investigate the growth of structure in the late universe. We will therefore content ourselves with using *CLASS*'s prediction for the linear matter power spectrum as our initial condition. Since evolution of the coupled system is linear, the statistical information encoded in the initial conditions is propagated forward in time without the introduction of additional nongaussianity. We therefore choose to fix our initial condition at $z = 0$ using the *CLASS* output, and instead evolve the system *backwards* in time. Figure (4.7) displays the initial conditions (at $z = 0$) for both a $\Lambda$CDM cosmology and an Einstein-de Sitter background universe. Given the linear solution up to $z = 0$, from an Einstein-Boltzmann code, we can then compute perturbative corrections at least back until the universe was radiation dominated.

### 4.3.2. Bispectrum

Whilst Wick's theorem implies that $\delta_1(k)$, the linear density contrast is completely described by its linear matter power spectrum, the situation is more complicated if we wish to consider the statistics of the full matter density contrast,

$$\delta = \delta_1 + \frac{1}{2}\delta_2 + \dots\ . \qquad (4.186)$$

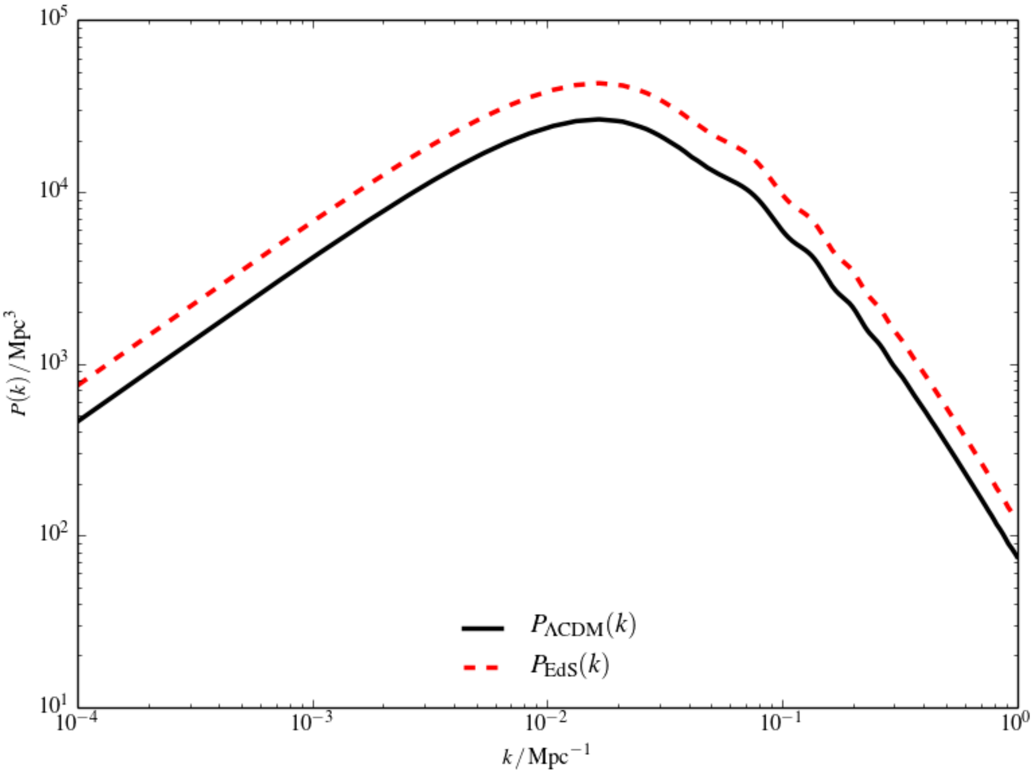


Figure 4.7.: A plot showing $P(k)$ as calculated by the *CLASS* Boltzmann code at $z = 0$ for both EdS and $\Lambda$CDM universes. Normalisation has been chosen so that $\mathcal{D}(z = 0) = 1$ in $\Lambda$CDM universes, whilst there is a small enhancement in EdS universes. *CLASS* performs its calculations in synchronous comoving gauge by default.

For example, whilst the leading order *bispectrum* (the Fourier transform of the 3-point correlation function) may vanish

$$\langle \delta_1(k_1)\delta_1(k_2)\delta_1(k_3)\rangle = 0 \,, \tag{4.187}$$

the full matter density contrast will have a non-zero bispectrum due to the interaction between the second order solution $\delta_2$ and two linear solutions $\delta_1$. This correction to the linear theory is the leading order correction induced by the second-order non-linear solution. We will illustrate this below in the Newtonian perturbation theory case (for an Einstein-de Sitter background):

$$(2\pi)^3 B(k_1,k_2,k_3)\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) = \langle \delta_N(\boldsymbol{k}_1)\delta_N(\boldsymbol{k}_2)\delta_N(\boldsymbol{k}_3)\rangle \; , \tag{4.188}$$

$$= \Big\langle \big(\delta_N^{(1)}(\boldsymbol{k}_1) + \frac{1}{2}\delta_N^{(2)}(\boldsymbol{k}_1) + \dots\big)\big(\delta_N^{(1)}(\boldsymbol{k}_2) + \frac{1}{2}\delta_N^{(2)}(\boldsymbol{k}_2) + \dots\big)\times \big(\delta_N^{(1)}(\boldsymbol{k}_3) + \frac{1}{2}\delta_N^{(2)}(\boldsymbol{k}_3) + \dots\big)\Big\rangle \; . \tag{4.189}$$

Since Wick's theorem implies that all odd correlators between gaussian random fields vanish, the leading order contribution must have four factors of $\delta^{(1)}(\boldsymbol{k})$, and is given by

$$\frac{1}{2}\langle \delta_N^{(1)}(\boldsymbol{k}_1)\delta_N^{(1)}(\boldsymbol{k}_2)\delta_N^{(2)}(\boldsymbol{k}_3)\rangle + 2 \text{ cycl. perms ...} = (2\pi)^3 B(k_1,k_2,k_3)\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) \; . \tag{4.190}$$

The subleading, or *1-loop* (where the number of loops is given by the number of momentum integrals in the final answer) contributions will then be those with six factors of $\delta^{(1)}(\boldsymbol{k})$, and so on. Now given,

$$\delta_N^{(1)}(\boldsymbol{k}) = a\; \delta^{(1)}(\boldsymbol{k}) \; , \tag{4.191}$$

$$\delta_N^{(2)}(\boldsymbol{k}) = a^2 \int \mathrm{d}^3q_1 \mathrm{d}^3q_2 \; \delta^{(1)}(\boldsymbol{q}_1)\; \delta^{(1)}(\boldsymbol{q}_2)\; \delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\; F_2^{(s)}(\boldsymbol{q}_1,\boldsymbol{q}_2) \; , \tag{4.192}$$

we can insert these solutions into Equation (4.190) and then use Wick's theorem to evaluate

$$\Big\langle \delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{q}_1)\delta^{(1)}(\boldsymbol{q}_2)\Big\rangle = \Big\langle \delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\Big\rangle\Big\langle \delta^{(1)}(\boldsymbol{q}_1)\delta^{(1)}(\boldsymbol{q}_2)\Big\rangle + \Big\langle \delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{q}_2)\Big\rangle\Big\langle \delta^{(1)}(\boldsymbol{q}_1)\delta^{(1)}(\boldsymbol{k}_2)\Big\rangle + \Big\langle \delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{q}_1)\Big\rangle\Big\langle \delta^{(1)}(\boldsymbol{q}_2)\delta^{(1)}(\boldsymbol{k}_2)\Big\rangle \; . \tag{4.193}$$

Using the definition of the linear matter power spectrum and evaluating the momentum integrals using the delta functions, we obtain

$$B(k_1,k_2,k_3) = F_2^{(s)}(\boldsymbol{k}_1,\boldsymbol{k}_2)P_L(k_1)P_L(k_2) + 2 \text{ cycl. perms ...} \; , \tag{4.194}$$

where we have discarded the "tadpole" term proportional to

$$\int \mathrm{d}^3q \; F_2^{(s)}(\boldsymbol{q},-\boldsymbol{q})\; P_L(q)\; P_L(k)\; \delta^{(3)}(\boldsymbol{k}) \; , \tag{4.195}$$

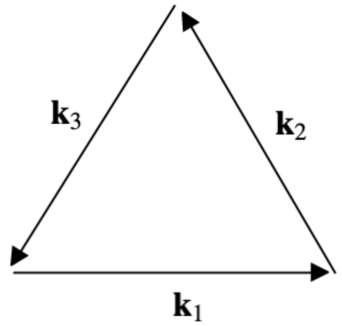


Figure 4.8.: The equilateral configuration.

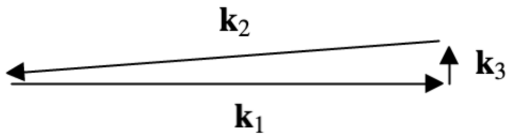


Figure 4.9.: The squeezed configuration.

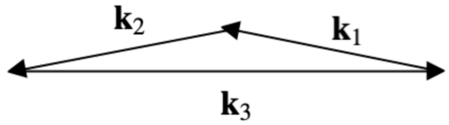


Figure 4.10.: The flattened configuration.

which is everywhere zero since $F_2^{(s)}(\boldsymbol{q}, -\boldsymbol{q}) = 0$ (apart from at the unphysical scale, $k = 0$, where it technically diverges).

The bispectrum is a function of three momenta, $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$, constrained to add up to zero in a triangle by the momentum-conserving delta function. Visualising the complex three-dimensional dependence of the bispectrum on various triangular configurations is therefore somewhat complicated. We will consider three specific limits, the *equilateral configuration*, where $B(k_1, k_2, k_3) = B(k, k, k)$, i.e the length of each vector in the triangle is the same, a moderately *squeezed configuration*, where $B(k_1, k_2, k_3) = B(k, k, k/16)$, where one of the lengths of the vectors in the triangle is taken to be much smaller than the other two, and a *flattened configuration*, where we consider a flattened iscoceles triangle with $k_1 = k_2 \sim \frac{1}{2}k_3$. The squeezed limit, in particular, is a natural arena in which to consider the coupling of large scales to small scales. It is important to remember however that these limits do not represent the full three-dimensional scale dependence of this function.

### Primordial nongaussianity

The bispectrum induced by gravity is a key signature of nonlinear evolution, since all odd correlators of the linear matter density contrast vanish under purely linear

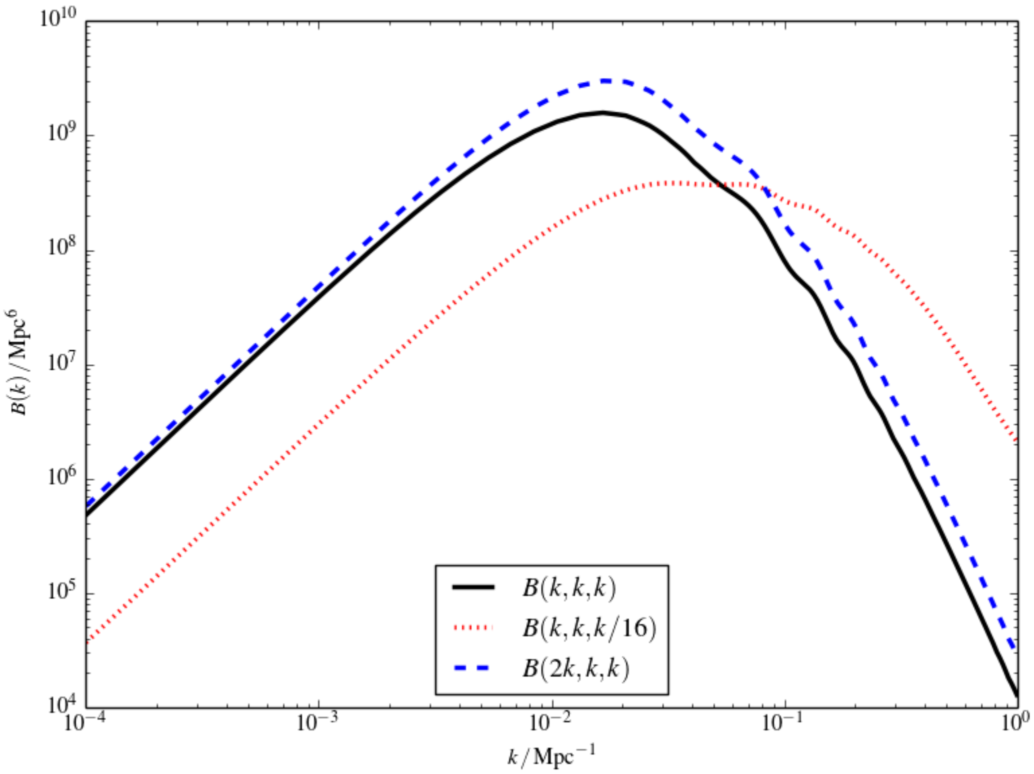


Figure 4.11.: Plots of the contributions from the diagrams in Figures (4.12). We show the $z = 0$ tree level Newtonian bispectrum induced by gravity for the equilateral configuration, $B_{\mathrm{Eq}} = B(k, k, k)$, a moderately squeezed configuration $B_{\mathrm{Sq}} = B(k, k, k/16)$, and a moderately flattened configuration, $B_{\mathrm{Fl}} = B(2k, k, k)$.

evolution. We have seen in the previous section how nonlinear evolution (even in the purely Newtonian case) automatically induces nongaussianity even given gaussian initial conditions. This type of nongaussianity is referred to as *induced nongaussianity* and is to be distinguished from *primordial nongaussianity* - nongaussianity that is generated by some unknown theory of inflation and that potentially contains information about new physics.

Naturally, detecting and constraining primordial nongaussianity is a huge science goal for upcoming surveys, since one might hope that some information gleaned would allow us to distinguish between the many different inflationary models [101–103]. However, in order to constrain primordial nongaussianity, it is necessary to understand the effect of its inclusion on the outcome of perturbative calculations of the type performed in the previous section. It would not be economical to calculate specific initial conditions outputted by every single model of inflation, and then recalculate the outcomes of perturbation theory. Instead, the amplitude of primordial nongaussianity is usually encoded in the parameter $f_{\rm nl}$:

$$\varphi_{\rm NG} = \varphi + f_{\rm nl}\big(\varphi^2 - \langle\varphi\varphi\rangle\big) \,, \tag{4.196}$$

where $f_{\rm nl}$ is the first coefficient in a Taylor series expansion away from the gaussian case described by $\varphi$. One can easily see that a non-gaussian field $\varphi_{\rm NG} = \varphi + f_{\rm nl}\big(\varphi^2 - \langle\varphi\varphi\rangle\big)$ will have a non-zero 3-point correlation function due to the presence of products of at least four gaussian random fields in certain terms proportional to $f_{\rm nl}$ when the full expression is expanded out. The inclusion of the effects of primordial nongaussianity into the scheme of perturbation theory is therefore facilitated by a choice of the value of $a_{\rm nl}$, the parameter appearing in the second-order initial conditions. In this case, choosing $a_{\rm nl} = 1$ is equivalent to choosing $f_{\rm nl} = 0$, i.e neglecting the effects of primordial nongaussianity, although the general scenario is more complex as was argued in [104].

For the remainder of this thesis, we will only consider Gaussian initial conditions when making plots; however, analytic results will be given for arbitrary $a_{\rm nl}$, thereby enabling studies constraining the value of this parameter.

### Diagrams

Calculations of this type can be expressed in a very elegant and intuitive fashion using Feynman diagram techniques borrowed from quantum field theory [86]. We will give a brief introduction to this shorthand, as it is an extremely useful time-saving device for dealing with the combinatorics involved in the application of Wick's theorem, especially when it comes to considering higher order corrections. Let us consider the problem of calculating the $r$-th order contribution to the general $n$-point function:

$$\big\langle Q_1(\boldsymbol{k}_1)\ldots Q_n(\boldsymbol{k}_n)\big\rangle \,, \tag{4.197}$$

where $Q_i$ can be any perturbation of either $\delta(\boldsymbol{k})$ or $\theta(\boldsymbol{k})$ depending on what statistic is under consideration. The Feynman rules that will directly output an expression for this quantity in terms of $P_L(k)$, $F_m^{(s)}$ and $G_m^{(s)}$ are:

- [i] Draw all connected diagrams containing $n$ vertices connected by $r$ internal lines. Each vertex will represent one of the perturbations $Q_i$. Two diagrams are distinct if they cannot be deformed into one another without cutting any internal lines. Sliding lines over other lines is allowed in the rearrangement procedure.

- [ii] Label each external line with an external momentum $\boldsymbol{k}_i$, where $i \in \{1, \dots, n\}$, and each internal line with an internal momentum, $\boldsymbol{q}_j$, where $j \in \{1, \dots, r\}$.

- [iii] To each vertex of order $m$, (where the *order* of the vertex is *CLASS*ified by the number of internal lines connected to it), associate a factor of $\delta^{(3)}(\boldsymbol{k}_i + \sum_j^k \boldsymbol{q}_j) K_m^{(s)}(\boldsymbol{q}_j, ..., \boldsymbol{q}_k)$, where $K_m^{(s)}$ is the integral kernel (either $F_m^{(s)}$ or $G_m^{(s)}$) relevant depending on which perturbation the vertex represents. For the signs of the internal momenta, $\boldsymbol{q}_i$, we use the convention that a positive sign indicates that the momenta is outgoing from the vertex.

- [iv] Assign a factor of $P_L(q_j)$ to each internal line.

- [v] Integrate over all $\boldsymbol{q}_j$.

- [vi] Multiply by the symmetry factor of the diagram (the number of ways in which you can permute the internal lines without altering the diagram).

- [vii] Multiply by $(2\pi)^3 \prod_i \frac{1}{m_i}$ to account for the factorials in the definition of the perturbations and the $2\pi$'s from the Fourier transforms.

We note that these Feynman rules are somewhat unusual compared to the familiar ones from quantum field theory, since new vertices appear at each order in perturbation theory rather than just increasing numbers of combinations of some fundamental vertices. This reduces their practical utility, since the main computational challenge is in the calculation of vertices (integral kernels) themselves. The methodology does however give a clear representation of the key statistics, and is a useful timesaving device.

For demonstrative purposes we will now apply this technique to calculate the tree level bispectrum, as we did before directly using Wick's theorem.

Consider the application of the Feynman rules to Figure (4.12). We immediately obtain the following expression:

$$\begin{aligned} \text{Diagram} = \frac{(2\pi)^3}{2} \cdot 2 \cdot \int \mathrm{d}^3 q_1 \mathrm{d}^3 q_2 \; \delta^{(3)}(\boldsymbol{k}_1 + \boldsymbol{q}_1 + \boldsymbol{q}_2) F_2^{(s)}(\boldsymbol{q}_1, \boldsymbol{q}_2) \times \\ \delta^{(3)}(\boldsymbol{k}_2 - \boldsymbol{q}_2) \delta^{(3)}(\boldsymbol{k}_3 - \boldsymbol{q}_1) P_L(q_1) P_L(q_2) \;, \end{aligned} \tag{4.198}$$

which, after performing the trivial delta function integrals and cyclically permuting the $\boldsymbol{k}_i$'s to take the other two diagrams into account, yields the same result as our

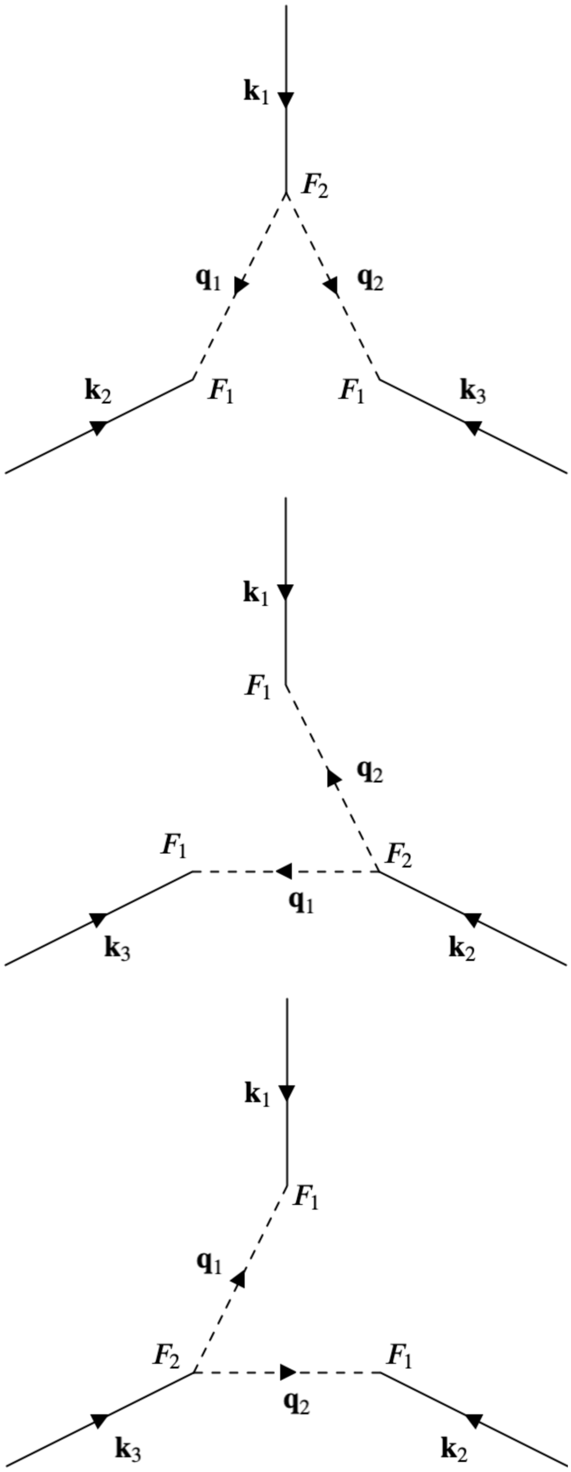


Figure 4.12.: The three tree level diagrams for the bispectrum, $\langle \delta_N(\boldsymbol{k}_1)\delta_N(\boldsymbol{k}_2)\delta_N(\boldsymbol{k}_3)\rangle$. Vertices have been labelled by the kernels they correspond to. The symmetry factors of these graphs is 2. One can obtain the other two diagrams by cyclically permuting the momenta associated with the external vertices on one initial diagram.

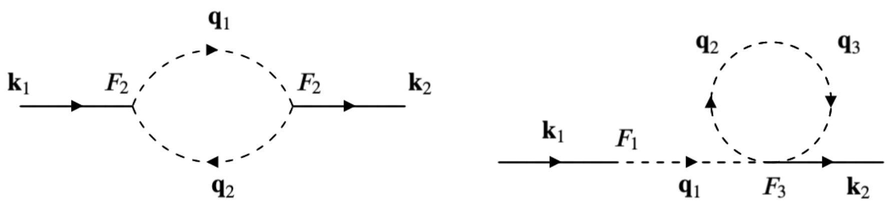


Figure 4.13.: The two distinct diagrams that contribute to the 1-loop power spectrum, $\left\langle \delta_N(\boldsymbol{k}_1)\delta_N(\boldsymbol{k}_2)\right\rangle_{1-\text{loop}}$. The symmetry factor of the graph on the left is 2, whilst the symmetry factor of the graph on the right is 6. Vertices have been labelled by the kernel they correspond to.

previous expression, (4.194),

$$\left\langle \delta_N(\boldsymbol{k}_1)\delta_N(\boldsymbol{k}_2)\delta_N(\boldsymbol{k}_3)\right\rangle = (2\pi)^3 F_2^{(s)}(\boldsymbol{k}_1,\boldsymbol{k}_2)P_L(k_1)P_L(k_2)\,\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) + 2\text{ cycl. perms} \ldots . \tag{4.199}$$

We can use the diagram technique to quickly evaluate some other quantities that we will be interested in later on. Consider the 1-loop corrections to the matter power spectrum. The two distinct diagrams are shown in Figure (4.13).

We can easily evaluate these two diagrams and compare to the definition of matter power spectrum to obtain the well known expressions:

$$P_{22}(k) = \int \mathrm{d}^3q\; P_L(q)P_L(\boldsymbol{k}-\boldsymbol{q})\left(F_2^{(s)}(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})\right)^2 , \tag{4.200}$$

$$P_{13}(k) = \int \mathrm{d}^3q\; P_L(q)P_L(k)F_3^{(s)}(\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k}) . \tag{4.201}$$

Although both these integrals contain divergences, (as $q \to 0$ and as $\boldsymbol{q} \to \boldsymbol{k}$ for $P_{22}$ and as $q \to 0$ for $P_{13}$) , it can be shown that the divergences in $P_{22}$ precisely cancel those in $P_{13}$ [105], as a direct consequence of the Galilean invariance of the theory [106]. Cancellations of this type are in fact guaranteed to happen at every order. We will subsequently find that this is no longer the case when we consider the relativistic generalisation of perturbation theory, and that the IR behaviour can be greatly affected by *gauge choices*. This is not unexpected however; nor is it problematic, since $P_L(k)$ is not truly expected to be a gauge-invariant quantity, and is not itself a directly observable quantity. To actually calculate observables, one would have to consider factoring in all manner of projection effects that arise from observation on the past lightcone, and include a bias model as is done in the calculation of the observed galaxy number counts in [107], [108] and [109].

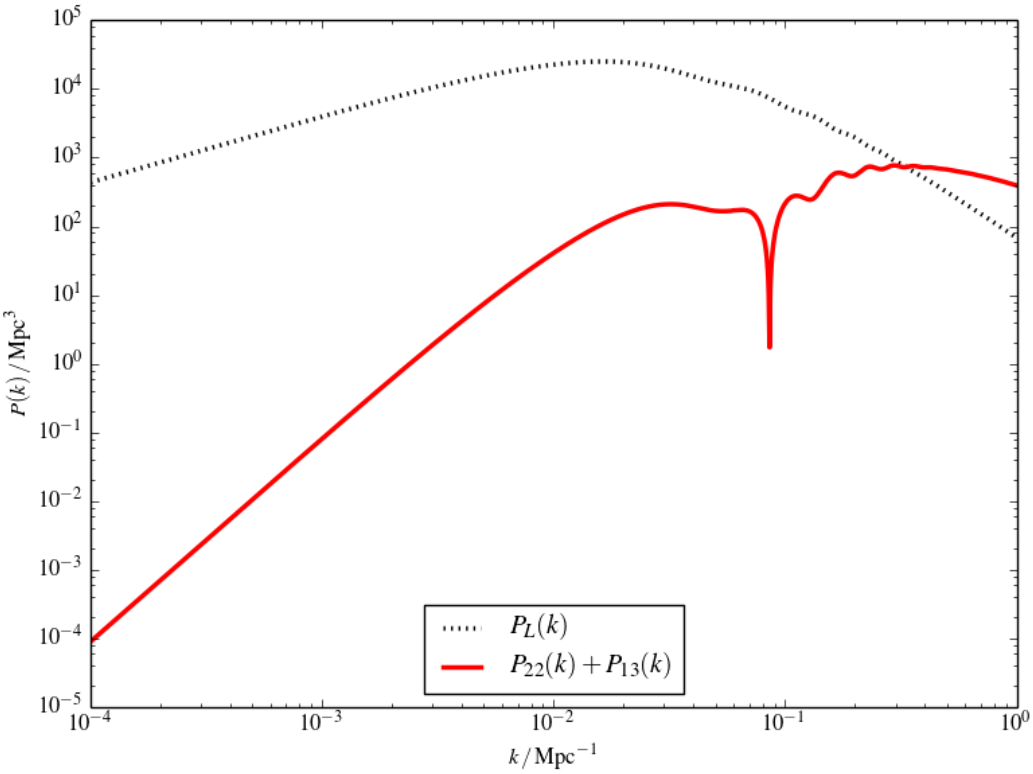


Figure 4.14.: Plots of the contributions from the diagrams in Figure (4.13). The panel shows plots of the absolute value of the 1-loop corrections to the power spectrum, alongside the linear matter power spectrum itself. The sign of the corrections changes from positive to negative at scales of around $k \sim 0.1\mathrm{Mpc}^{-1}$. Here $P_{11} = P_L$ by definition.

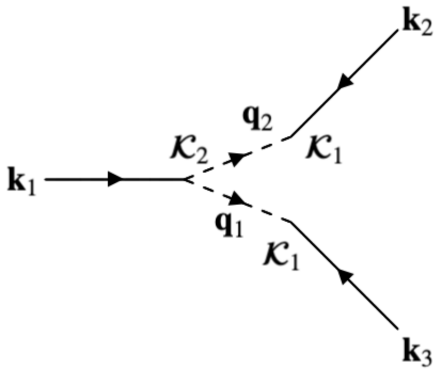


Figure 4.15.: One of the diagrams for the tree level relativistic bispectrum, $\langle \delta_1(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\delta_2(\boldsymbol{k}_3)\rangle$.

A variety of numerical techniques are then available for numerically evaluating the so called *IR-safe* integrals that result, such as those implemented in the *FAST-PT* code [106]. Calculations in this thesis requiring such integration routines were performed using the *FAST-PT* code.

### 4.3.3. Relativistic statistics

We have already mentioned a fundamental problem with calculating quantities such as $P_L(k)$ in the context of general relativity - such quantities are not directly observable and are in fact highly gauge dependent (especially at large scales). Therefore the usual practice is to try to calculate the lightcone-projected versions of these quantities [107] [108] [109] [110]. Such calculations can be notoriously involved though, and for the moment we will restrict ourselves to the calculation of gauge dependent statistics, since for our purpose of comparing two-parameter results to second order perturbation theory, the calculation of the Poisson gauge statistics will be sufficient. We will again focus on the diagram, this time modifying the Feynman rules appropriately such that at vertices, we use fully relativistic kernels, $\mathcal{K}_n$ instead of Newtonian ones, and that each internal line comes with a factor of the Poisson gauge linear power spectra. We can immediately write down the following expression;

$$B(k_1, k_2, k_3) = \mathcal{K}_2^{\delta}(\boldsymbol{k}_1, \boldsymbol{k}_2) P_p(k_1) P_p(k_2) + \text{ 2 cycl. perms} \dots \ , \tag{4.202}$$

where $B(k_1, k_2, k_3)$ is the tree level bispectrum evaluated in Poisson gauge, $P_p(k)$ is the linear matter power spectrum evaluated in Poisson gauge, and $\mathcal{K}_2^{\delta}(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is the

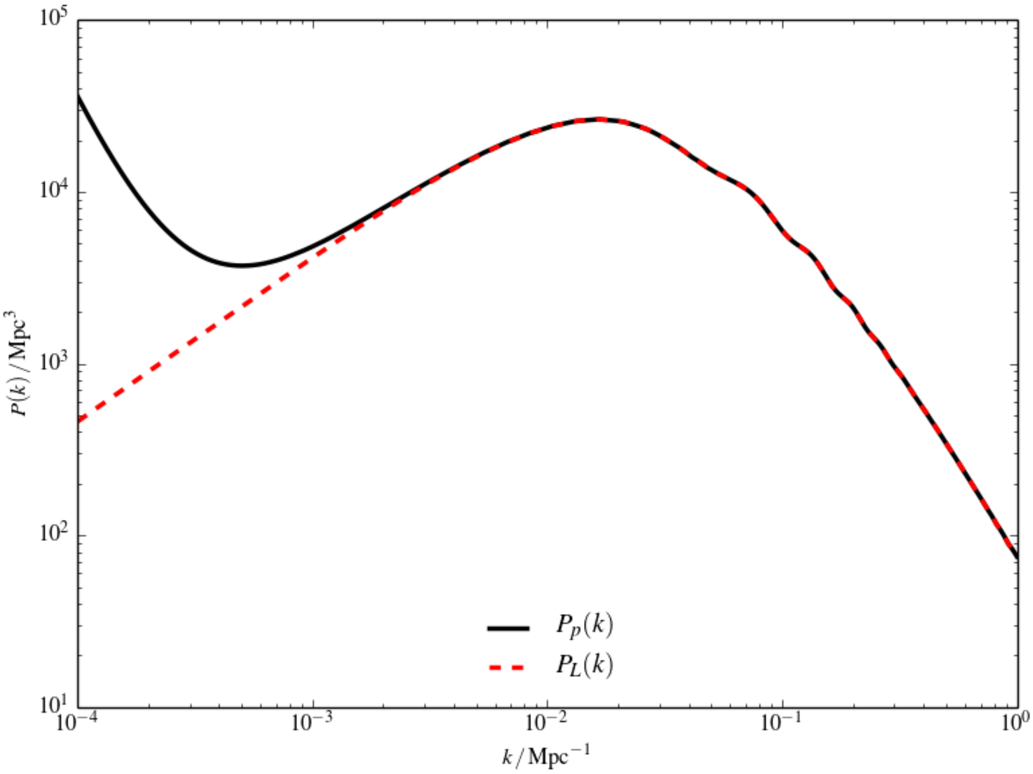


Figure 4.16.: A plot of the linear matter power spectrum predicted by *CLASS*, evaluated in Poisson gauge (red line) and synchronous comoving gauge (black line).

second order relativistic matter density kernel, defined by

$$\delta_2(\boldsymbol{k},\tau) = \int \mathrm{d}^3q_1\mathrm{d}^3q_2\ \mathcal{K}_2^{\delta}(\boldsymbol{q}_1,\boldsymbol{q}_2,\tau)\ \delta_1(\boldsymbol{q}_1,\tau)\ \delta_1(\boldsymbol{q}_2,\tau)\ , \tag{4.203}$$

wherein we have included time dependence in both the kernel and the linear matter density contrasts, since the spatial and time dependencies are no longer separable. An explicit expression for the relativistic kernel in the $\Lambda$CDM universe was calculated in the previous section, specifically Equation (4.149). The Einstein-de Sitter limit of this expression can be found in the usual way.

We plot the bispectrum at $z = 0$ in the equilateral configuration in Figure 4.17. The typical effects of the inclusion of the cosmological constant can be seen, namely a damping of the overall amplitude, and also a damping of the relativistic corrections. In the Einstein-de Sitter background cosmology, the bispectrum possesses two zero crossings, induced by relativistic corrections whereas it can be seen that this feature does not occur in the $\Lambda$CDM version. We can further see the effects of inclusion of the cosmological constant in the squeezed limit of the bispectrum,

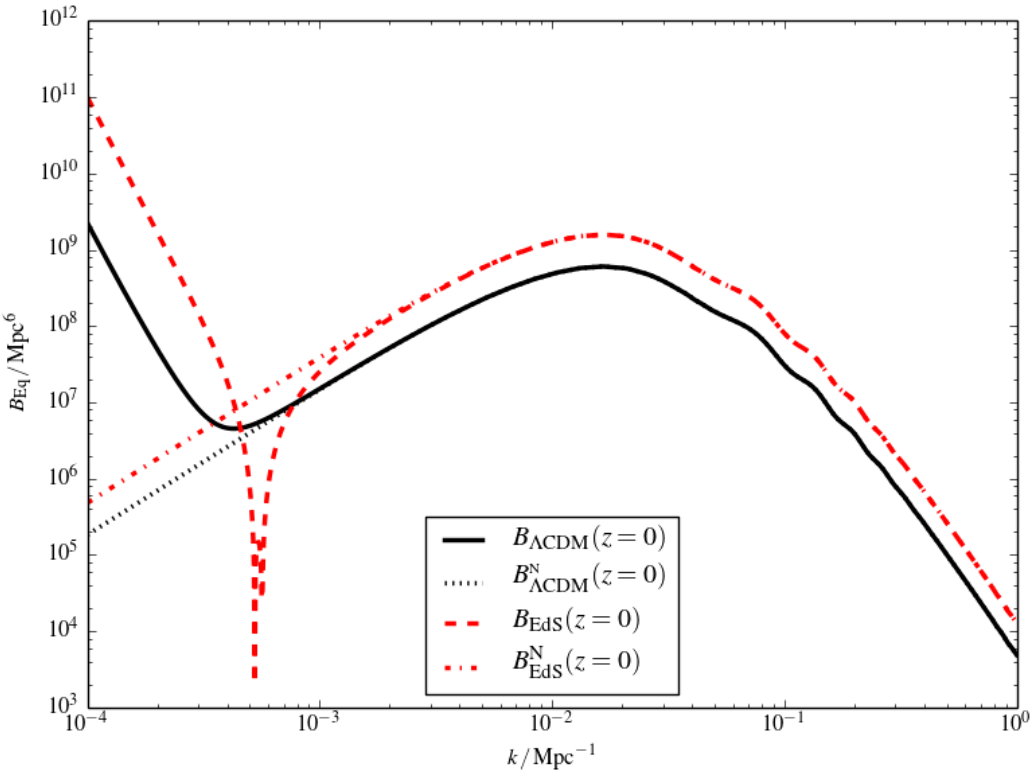


Figure 4.17.: A plot of the relativistic bispectrum in the equilateral configuration at $z = 0$, for both $\Lambda$CDM and Einstein-de Sitter background cosmologies.

plotted at redshift $z = 0$ in Figure 4.18. It is patently clear that the magnitude of relativistic corrections is significantly larger in the Einstein-de Sitter background geometry, with relativistic corrections also becoming noticeable at much smaller scales. Zero crossings are present for both $\Lambda$CDM and Einstein-de Sitter backgrounds. The inclusion of the cosmological constant also has a significant effect on the flattened configuration of the bispectrum of matter, which is plotted in both the $\Lambda$CDM and Einstein-de Sitter cases in Figure 4.19. In a similar fashion to the equilateral case, the Einstein-de Sitter relativistic corrections have two zero crossings, whilst the $\Lambda$CDM crossings do not. In order to examine the time evolution of these quantities, we plot the $\Lambda$CDM bispectra at redshifts of $z = 0$, $z = 0.5$,$z = 1$, and $= 5$ in each configuration in Figures 4.20, 4.21, and 4.22. As expected, the overall amplitude of the bispectrum decreases with redshift, but the scales at which relativistic corrections become important also decrease, due to the decrease in size of the horizon with redshift. The expressions for the $\Lambda$CDM integral kernel have been checked carefully by plotting them in the limit as $\Omega_{m0} = 1$ directly against the expressions for the Einstein-de Sitter kernel obtained by analytically taking the limit. Kernels coincide in all cases.

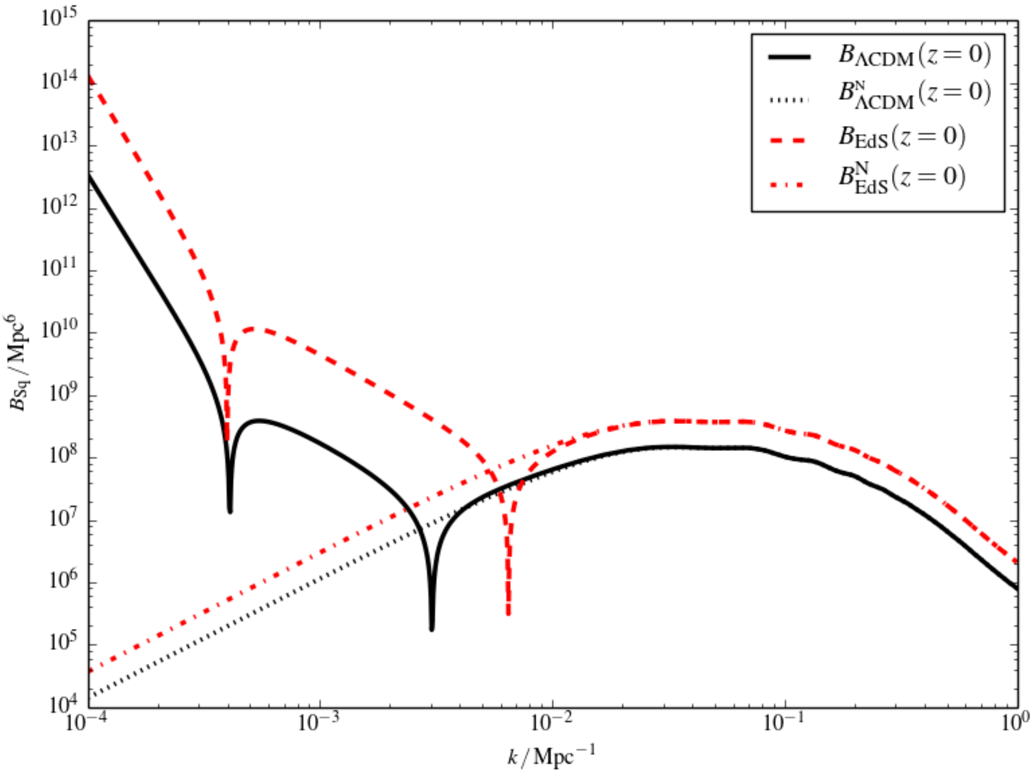


Figure 4.18.: A plot of the relativistic bispectrum in the squeezed configuration at $z = 0$, for both ΛCDM and Einstein-de Sitter background cosmologies. Here, $k_1 = k_2 = k$ and $k_3 = \frac{k}{16}$.

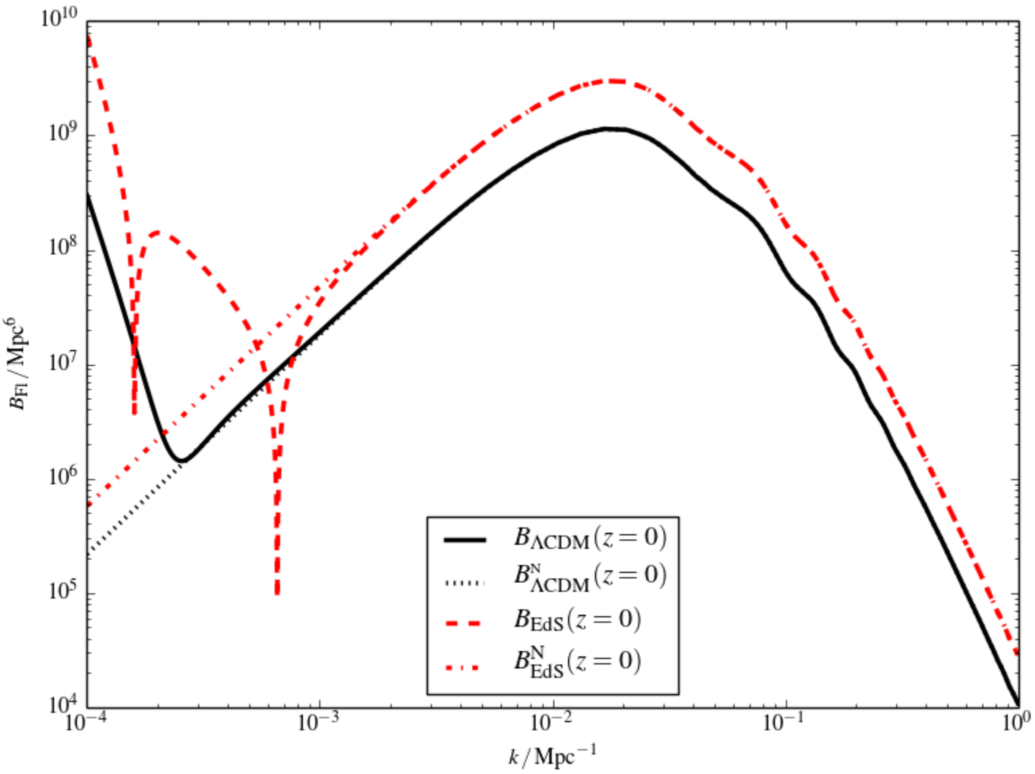


Figure 4.19.: A plot of the relativistic bispectrum in the flattened configuration at $z = 0$, for both $\Lambda$CDM and Einstein-de Sitter background cosmologies. Here, $k_1 = 2k$, $k_2 = k_3 = k$.

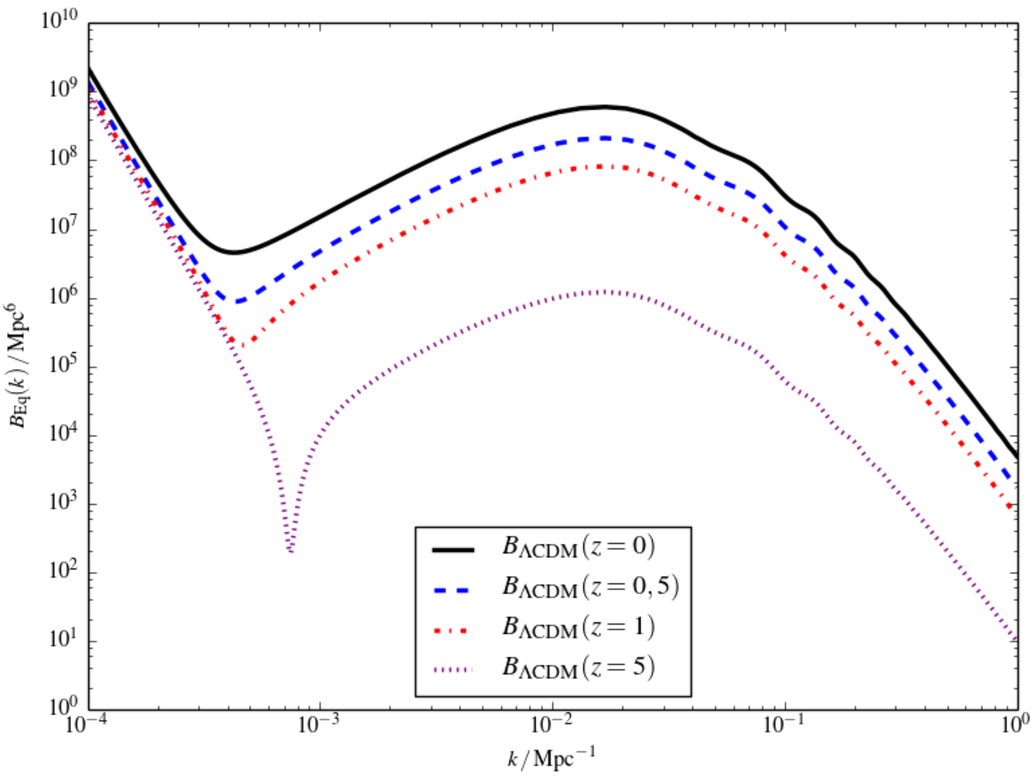


Figure 4.20.: A plot of the relativistic $\Lambda$CDM bispectrum in the equilateral configuration at $z = 0$, $z = 0.5$ $z = 1$ and $z = 5$. Here, $k_1 = k_2 = k_3$.

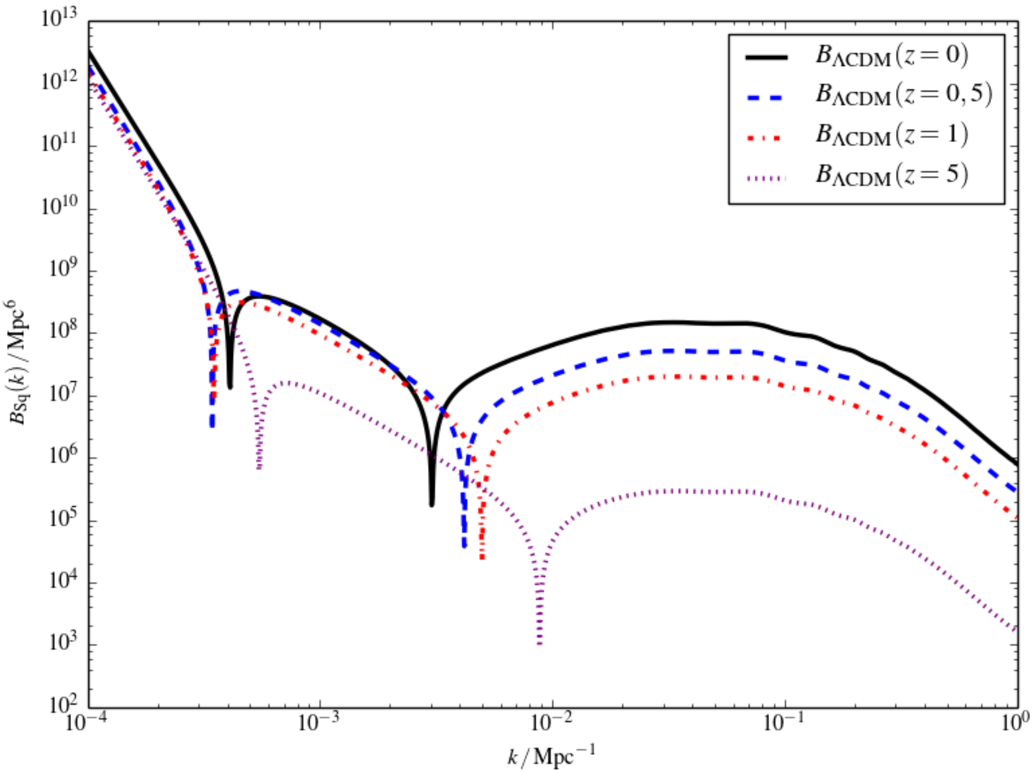


Figure 4.21.: A plot of the relativistic ΛCDM bispectrum in the squeezed configuration at $z = 0$, $z = 0.5$ $z = 1$ and $z = 5$. Here, $k_1 = k_2 = k$ and $k_3 = \frac{k}{16}$.

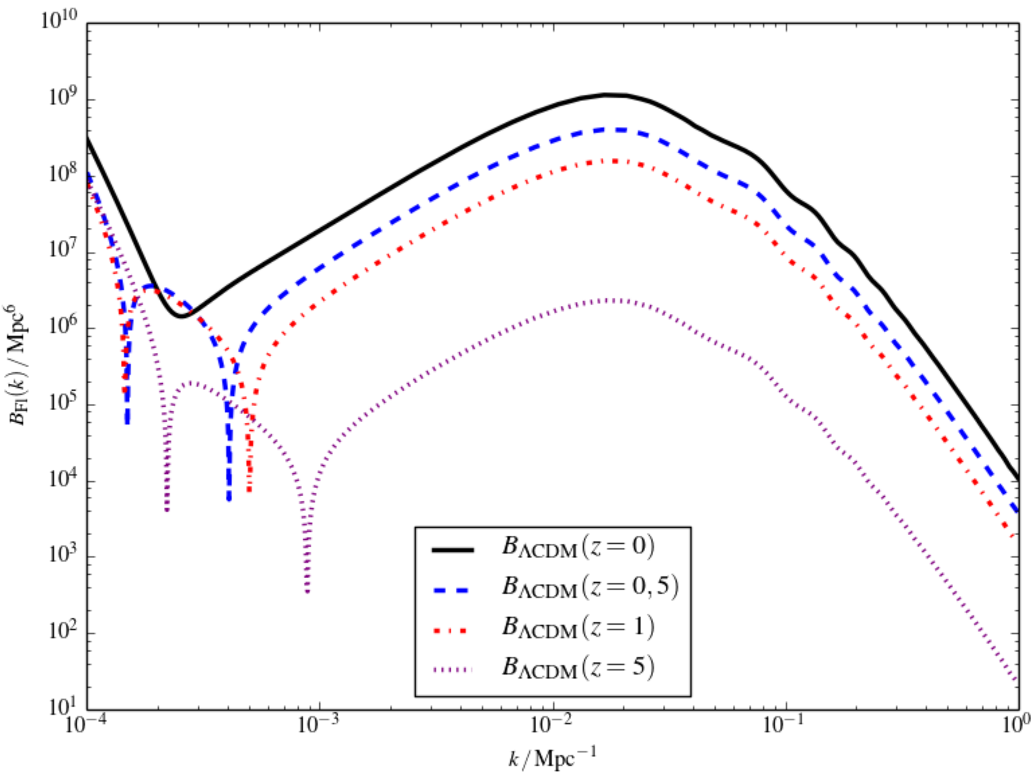


Figure 4.22.: A plot of the relativistic ΛCDM bispectrum in the flattened configuration at $z = 0$, $z = 0.5$ $z = 1$ and $z = 5$.

# 5. Viable Gauge Choices in Cosmologies with Nonlinear Structure

The work presented in this chapter is based on the paper [80]. The primary observations about the viability of various gauges were made by Timothy Clifton, following his work with Sophia Goldberg. Karim Malik provided useful information and perspective on cosmological perturbation theory. Higher order post-Newtonian calculations were checked by myself, by modifying the Mathematica package "xPand" to deal with post-Newtonian expansions on a Friedmann background [111].

## 5.1. Motivation

Upcoming galaxy surveys such as Euclid, SKA and DESI will complement the Planck CMB survey by extending direct probes of the matter distribution in the late universe up to almost the scale of the horizon. When theoretically modelling the physics of the anisotropy of the microwave background, the results of the cosmological perturbation theory developed in the previous chapter can generally be taken to provide an excellent description on all scales. This is because the CMB is comprised of photons travelling on null geodesics from the time of decoupling and as such, are not susceptible to gravitational collapse (although they are affected by the integrated Sachs-Wolfe effect and reionisation). The temperature of the CMB is related to the density perturbation at the time of decoupling, when the fluctuations are extremely well modelled by linear perturbation theory. Nonrelativistic dark matter, the dominant form of matter in the universe according to the ΛCDM model, moves slowly however and is therefore much more susceptible to the nonlinear effects of gravity. Accordingly, in the late universe where significant nonlinear structures have formed ($z < 10$), one is liable to find density contrasts of order $\delta \sim \mathcal{O}(1)$, at least on small spatial scales, $L_N < 100$ Mpc.

These nonlinear structures manifest in the form of virialised *superclusters*, vast structures of galaxies that are gravitationally bound, but are low density enough to still expand with the Hubble flow. Superclusters themselves have been found to exhibit a rich phenomenology, encompassing the so-called “cosmic web” of ”sheets” and “filaments”, interspersed with “voids”. They may individually span scales of up to $\sim 100$ Mpc, and are the largest known gravitationally bound structures in the universe.

The typical approximation scheme used to model the physics of these nonlinear structures is that of the post-Newtonian expansion, described in Section 3.2. As noted before, this approximation scheme has a very different mathematical structure to the typical cosmological perturbation theory expansions used to calculate observables for the cosmic microwave background; in particular, it possesses a different formulation of the gauge problem. By investigating the structure of the gauge problem in both cosmological perturbation theory and the post-Newtonian expansion, it is possible to identify gauge choices that are viable on the horizon-sized scales that cosmological perturbation theory is valid on, and also on the supercluster length scales $L_N < 100$ Mpc. It is also possible to identify cases where a typical gauge choice made in cosmological perturbation theory turns out to be inconsistent with the post-Newtonian expansion, and vice versa. In these cases, this indicates that these gauge choices are inappropriate for modelling cosmologies including nonlinear structure, since one expects the nonlinear structure to be accurately modelled by post-Newtonian expansions on scales of less than $L_N <$ 100Mpc.

## 5.2. Standard Gauge Choices in Cosmology

Choosing a gauge is often essential in cosmology. Gauge differences on large scales may be extremely important for a number of science goals, especially those involving constraints on primordial nongaussianity [112]. In order to relate the actual observed overdensity on the lightcone to the overdensity in a particular gauge, GR corrections must be considered to take both gauge effects and lightcone effects into account. The definition of bias is also most naturally carried out in comoving gauges, and its definition can therefore be subject to gauge dependencies. These effects can sometimes mimic the effect of local type primordial nongaussianity characterised by $f_{\mathrm{NL}}$, therefore rendering a proper understanding essential in order to disentangle the effects of these distinct physical phenomena and optimise constraints on $f_{\mathrm{NL}}$.

Unfortunately, the majority of gauges that are frequently used in the literature are not viable choices in the presence of non-linear structures modelled by post-Newtonian theory. In this section we will review some of the "popular" gauges used in cosmological perturbation theory (see e.g. Ref. [31] for details). These gauges are usually specified by assigning a particular set of variables to zero, either in the gravitational sector or the matter sector (or a mixture of both), by making specific choices for the gauge generators in the gauge transformation equations for the metric, Equations (4.41) and for matter, Equations (4.45). In each case we will also comment on whether such a gauge can be achieved in the post-Newtonian expansion by making equivalent choices for the post-Newtonian gauge generators in Equations (3.66) and Equations (3.77).

### 5.2.1. Spatially Flat Gauge

The spatially flat gauge is defined by the choice

$$\psi = E = F_i = 0\,, \tag{5.1}$$

which leaves the induced 3-metric on spatial hypersurfaces unperturbed (in the absence of tensor perturbations). This gauge is often used for the calculation of observables during inflation.

It can be seen from Eq. (4.41) that this gauge can be readily achieved in cosmological perturbation theory by choosing $\xi^0 = \psi/\mathcal{H}$, and $\xi_i = -E_{,i} - F_i$. On the other hand, in the post-Newtonian theory $\psi$ is gauge-invariant, and so this gauge is impossible to realise (though it is possible to set $E$ and $F_i$ to zero).

### 5.2.2. Synchronous Gauge

Synchronous gauge is defined by setting

$$\phi = B = S_i = 0\,. \tag{5.2}$$

This gauge is popular for numerical studies, but does not uniquely define the time-slicing (this can be fixed by choosing an additional gauge condition, for example that the perturbed dark matter 3-velocity vanishes). In this gauge it can be seen that the time coordinate corresponds to the proper time of comoving observers at fixed spatial coordinates. Synchronous gauge is routinely used in a wide variety of cosmological calculations, and is the default gauge for CMBFAST [113], CAMB [114] and *CLASS* [33].

This gauge is obtained within cosmological perturbation theory by solving the differential equations $\xi^{0\prime} + \mathcal{H}\xi^0 = -\phi$ and $\xi_i' - \xi^0 = -B_{,i} + S_i$. However, it cannot be achieved in post-Newtonian theory as in this case $\phi$ is gauge invariant at leading order (though $B$ and $S_i$ are not).

### 5.2.3. Comoving Orthogonal Gauge

The comoving orthogonal gauge is defined by the gauge conditions

$$v_i = 0 \qquad \text{and} \qquad B_{,i} = S_i\,, \tag{5.3}$$

which states that the fluid 3-velocity and 3-momentum vanish. In this gauge the constant time hypersurfaces are orthogonal to the fluid 4-velocity. In cosmological perturbation theory this gauge choice requires $\xi^{i\prime} = v^i$ and $\xi^0_{,i} = \xi_i'$. Once more, this gauge choice cannot be realised in post-Newtonian theory, this time because $v^i$ is gauge invariant at leading order (though $B_{,i}$ and $S_i$ are not, and could be set equal).

### 5.2.4. Total Matter Gauge

The total matter gauge is related to the comoving orthogonal gauge. It has the gauge conditions

$$v + B = 0 \qquad \text{and} \qquad E = 0 = F_i\,. \tag{5.4}$$

Evaluating the density contrast in the total matter gauge, and the metric potential $\phi$ in the longitudinal gauge, allows one to write the cosmological perturbation equations in the form of a Poisson equation, equivalent to its Newtonian counterpart [115, 116]. This gauge can be realised in cosmological perturbation theory by choosing $\xi^0 = v + B$ and $\xi_i = -E_{,i} - F_i$. It cannot be realised in post-Newtonian theory as the condition $v + B = 0$ has parts at order $\eta$ and $\eta^3$, the former of which cannot be satisfied as it corresponds to $v = 0$, and $v$ is gauge invariant (though the other conditions are again possible).

### 5.2.5. Uniform Density Gauge

In the uniform density gauge we use the density perturbation, or equivalently the density contrast, to specify the temporal gauge condition

$$\delta\rho = 0\,. \tag{5.5}$$

To fix the spatial gauge we can choose, for example, $E = 0 = F_i$. In cosmological perturbation theory this choice of specification of the temporal gauge can be written as $\xi^0 = -\delta\bar{\rho}/\bar{\rho}'$, but such a condition is impossible to implement in the post-Newtonian approach as $\mu$ is gauge invariant in this set-up.

### 5.2.6. N-body gauge

The $N$-body gauge is formulated in a situation where

$$v + B = 0\,, \tag{5.6}$$

as in the total matter gauge, above. The remaining gauge freedoms are then used to set the so-called "counting density" associated with $N$ bodies equal to the leading-order part of the energy density. This condition requires that the scalar deformation of the spatial volume is set to zero, which can be written as [117]

$$\psi + \frac{1}{3}\nabla^2 E = 0\,. \tag{5.7}$$

This can be achieved in cosmological perturbation theory by taking $\xi^0 = v + B$ and setting the spatial gauge using the solution of $\nabla^2\zeta = 3\mathcal{H}(v+B) - \nabla^2 E - 3\psi$. Now, while $v + B = 0$ still cannot be realised in post-Newtonian gravity, the condition given in Eq. (5.7) can be achieved by taking $\nabla^2\zeta = -\nabla^2 E - 3\psi$. It may therefore be possible to develop new variants of the $N$-body gauge with alternative specification of the temporal gauge condition, such as the N-boisson gauge [118, 119].

### 5.2.7. Longitudinal Gauge

Longitudinal gauge (also referred to as the *conformal Newtonian* gauge, or as part of the *Poisson* gauge) is defined by the scalar gauge conditions

$$B = E = 0\,. \tag{5.8}$$

As the scalar shear is given by $\sigma = E' - B$, this gauge is also known as "zero-shear" gauge (the spatial hypersurfaces have vanishing shear). These gauge conditions give a diagonal metric tensor for the scalar perturbations, which considerably simplifies calculations. If there is no anisotropic stress, the field equations in this gauge give $\psi = \phi$, which allows one to write the governing field equations from cosmological perturbation theory in a form that is very close to the Newtonian equation. This gauge choice can also be fully implemented in post-Newtonian theory, as well as cosmological perturbation theory. This can be achieved in both cases by taking $\xi^0 = B + E'$ and $\zeta = -E$. As it is allowed in both types of weak-field expansion, this gauge choice therefore appears to be particularly valuable if one wishes to perform calculations in both the linear and non-linear regimes of cosmology, and to find results in each case that can be consistently related to one another.

In the next section we will consider "Newtonian motion gauge", which can also be applied in both types of weak field expansion.

## 5.3. Newtonian Motion Gauge

The *Newtonian motion gauge* was recently introduced by Fidler *et al* in Ref. [120], and was further developed in Ref. [121]. It is based on the idea of fixing a gauge such that the gravitational field equation and equations of motion of test particles take the same form that they do in the Newtonian problem, i.e. such that

$$\tilde{\mu}' + 3\mathcal{H}\tilde{\mu} + \partial_i\left(\tilde{\mu}\tilde{v}^i\right) = 0 \tag{5.9}$$

$$\tilde{\mu}\tilde{v}_j' + \tilde{\mu}\tilde{v}^i\partial_i\tilde{v}_j + \tilde{\mu}\mathcal{H}\tilde{v}_j = -\tilde{\mu}\partial_j\tilde{U} - \partial_j\tilde{P}\,, \tag{5.10}$$

where $\tilde{U}$ must satisfy an equation of the form

$$\nabla^2\tilde{U} = 4\pi\,\delta\tilde{\mu}\,a^2\,. \tag{5.11}$$

The variables $\tilde{v}^i$, $\tilde{\mu}$, $\tilde{U}$ and $\tilde{P}$ can be seen to satisfy equations of exactly the same form as the Newtonian equations, but are not themselves the Newtonian variables. Instead, they should be thought of as variables that are constructed from objects that are defined in the corresponding relativistic problem.

This is a very interesting idea, as almost all N-body simulations are based on the equations that result from considering Newtonian physics on an expanding background. The Newtonian motion gauge therefore allows Newtonian N-body simulations to be interpreted in a relativistic context, and therefore for relativistic gravitational effects to be extracted from non-relativistic simulations. This is achieved by deforming the coordinate system (using gauge transformations) such that the coordinate positions of particles are the same as those that would appear in the Newtonian problem. Here we will investigate this idea in the context of cosmological perturbation theory and post-Newtonian theory.

### 5.3.1. Cosmological Perturbation Theory

It is clear that the non-linear equations (5.9) and (5.10) will not be able to be satisfied by the linearised equations of first-order scalar cosmological perturbation theory (which we give here without specifying a gauge):

$$\nabla^2\psi - 3\mathcal{H}\left(\psi' + \mathcal{H}\phi\right) + \mathcal{H}\nabla^2\sigma = 4\pi a^2\delta\rho \tag{5.12}$$

$$\psi' + \mathcal{H}\phi = -4\pi a^2(\bar{\rho} + \bar{P})\left(v + B\right) \tag{5.13}$$

$$\psi'' + \mathcal{H}\left(2\psi' + \phi'\right) + \left(2\mathcal{H}' + \mathcal{H}^2\right)\phi = 4\pi a^2\delta P \tag{5.14}$$

$$\sigma' + 2\mathcal{H}\sigma - \phi + \psi = 0\,. \tag{5.15}$$

In order to establish whether or not this gauge is viable in such an approach, we therefore propose to expand Eqs. (5.9)-(5.10) perturbatively, and see whether or not the equations of cosmological perturbation theory can be manipulated into the form of the equations that result.

We start by writing

$$\tilde{\mu} = \bar{\tilde{\mu}} + \delta\tilde{\mu} + O(\epsilon^2) \tag{5.16}$$

$$\tilde{v}^i = \delta\tilde{v}^i + O(\epsilon^2)\,. \tag{5.17}$$

To background order we find that Eq. (5.9) can be written as

$$\bar{\tilde{\mu}}' + 3\mathcal{H}\,\bar{\tilde{\mu}} = 0\,, \tag{5.18}$$

which is clearly of the same form as the energy conservation equation, as long as $\bar{p} = 0$, whilst the momentum conservation equation (5.10) is automatically satisfied. We therefore have $\bar{\tilde{\mu}} = \bar{\rho}$, and the requirement $\bar{P} = 0$ (i.e. that we consider dust, at the level of the background).

Next, we can study the perturbed equations at first order. For Eqs. (5.9)-(5.10) this gives

$$\delta\tilde{\mu}' + 3\mathcal{H}\,\delta\tilde{\mu} + \tilde{\bar{\mu}}\,\delta\tilde{v}^i{}_{,i} = 0 \tag{5.19}$$

$$\tilde{\bar{\mu}}\,\delta\tilde{v}^{j\prime} + \tilde{\bar{\mu}}\mathcal{H}\,\delta\tilde{v}^j = -\tilde{\bar{\mu}}\,\tilde{U}_{,j} - \delta\tilde{P}_{,j}\,. \tag{5.20}$$

If we now consider the equation of energy conservation at first order in cosmological perturbation theory,

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) = (\bar{\rho} + \bar{P})\left[3\psi' - \nabla^2\left(v + E'\right)\right] \tag{5.21}$$

then we see that if we choose $\delta\tilde{\mu} = \delta\rho - 3\bar{\rho}\psi + \bar{\rho}\nabla^2 E$ and $\delta\tilde{v}^i = v^i$ then we can write this equation in the form of the linearised Newtonian equation (5.19). This gives us

$$\tilde{\mu} = \bar{\rho} + \delta\rho - 3\bar{\rho}\,\psi + \bar{\rho}\,\nabla^2 E + O(\epsilon^2) \tag{5.22}$$

and

$$\tilde{v} = v + O(\epsilon^2)\,, \tag{5.23}$$

where $\tilde{v}$ and $v$ are the scalar parts of $\tilde{v}^i$ and $v^i$, respectively. For this correspondence to follow we also require $\delta P = 0$ (i.e. that the requirement to consider dust is extended to linear order).

The combination of variables used to construct $\tilde{\mu}$ and $\tilde{v}$ in Eqs. (5.22) and (5.23) have not yet required any choice of gauge. Let us now consider the linearised momentum conservation equation (5.20) that these variables must satisfy. Substituting in from Eq. (5.23), and taking $\delta\tilde{P} = 0$, we find that the following equation must be satisfied:

$$B' + \mathcal{H}\,B = \tilde{U} - \phi\,, \tag{5.24}$$

where $\tilde{U}$ must now satisfy

$$\nabla^2\tilde{U} = 4\pi\,a^2\,\bar{\rho}\,(\delta - 3\psi + \nabla^2 E)\,. \tag{5.25}$$

The derivation of this equation has used the Euler equation from cosmological perturbation theory,

$$\partial_\tau\left[(\bar{\rho} + \bar{P})(v + B)\right] + \delta p = -(\bar{\rho} + \bar{P})\left[\phi + 4\mathcal{H}(v + B)\right]\,, \tag{5.26}$$

in order to eliminate $v'$, and can be seen to be equivalent to Eq. (4.5) of Ref. [121] (though without specifying any restriction on the time gauge).

Further manipulation, using the linear equations from cosmological perturbation theory with $\delta P = 0$, allows us to re-write Eq. (5.24) as

$$\boxed{E'' + \mathcal{H}E' - 4\pi\, a^2\, \bar{\rho}E = 3\,\bar{\rho}\,\Phi_{\mathcal{R}}}\,, \tag{5.27}$$

where

$$\Phi_{\mathcal{R}} = -a^2(\tau)\int \frac{\mathcal{R}(\hat{x})}{|\boldsymbol{x} - \hat{x}|} d^3\hat{x}\ , \tag{5.28}$$

and where $\mathcal{R} = \psi - \mathcal{H}(v + B)$ is the curvature perturbation in comoving orthogonal gauge (a well-known gauge invariant quantity, frequently used in cosmology). The boxed equation in (5.27) needs to be satisfied if the Newtonian motion gauge is to be realised in cosmological perturbation theory.

We should point out that the choices for the effective Newtonian variables made in Eqs. (5.22) and (5.23) are not unique, though they did lead to a viable application of the idea of a Newtonian motion gauge. Instead of Eqs. (5.22) and (5.23), we could have equally well chosen our effective Newtonian variables to be

$$\tilde{\mu} = \bar{\rho} + \delta\rho - 3\bar{\rho}\psi + O(\epsilon^2) \tag{5.29}$$

and

$$\tilde{v} = v + E' + O(\epsilon^2)\,, \tag{5.30}$$

which would have also satisfied the linearized Newtonian equation of energy conservation (5.19). Substituting into the linearised momentum conservation equation (5.20) from Eq. (5.30), and taking $\delta\tilde{P} = 0$, we find that in this case the following equation must be satisfied:

$$v' + \mathcal{H}\,v + E'' + \mathcal{H}\,E' = -\hat{U}\,, \tag{5.31}$$

where $\hat{U}$ is

$$\nabla^2\hat{U} = 4\pi\, a^2\, \bar{\rho}\,(\delta - 3\psi)\,. \tag{5.32}$$

This can be equivalently written as

$$\sigma' + \mathcal{H}\,\sigma = \phi - \hat{U}\,, \tag{5.33}$$

where $\sigma = E' - B$. This equation needs to be satisfied if the Newtonian momentum conservation equation is to be true for the variables in Eqs. (5.29)-(5.30).

We can now use the evolution equation for $\sigma$, Equation (5.27), to find that the condition in Eq. (5.33) is equivalent to requiring

$$\mathcal{H}\,\sigma = \hat{U} - \psi\,. \tag{5.34}$$

In order to evaluate this equation, we can use Eqs. (5.12) and (5.32) to write

$$\nabla^2(\tilde{U} - \psi) = \mathcal{H}\,\nabla^2\sigma - 12\pi\,a^2\,\bar{\rho}\,(\psi - \mathcal{H}(v+B))\;. \tag{5.35}$$

This equation makes it clear that Eq. (5.34) is satisfied for $\bar{\rho} \neq 0$ if and only if

$$\psi - \mathcal{H}(v+B) = 0\,, \tag{5.36}$$

where sensible boundary conditions have been assumed.

One may now note that the combination of variables on the left-hand side of Eq. (5.36) is equal to the curvature perturbation in comoving orthogonal gauge, $\mathcal{R} = \psi - \mathcal{H}(v+B)$, a well-known gauge invariant quantity. It is therefore impossible to satisfy Eq. (5.36), and hence Eq. (5.20), by a choice of gauge using the variables in Eqs. (5.29) and (5.30). This shows that the choice of effective variables is extremely important in the implementation of this gauge, and that although the Newtonian motion gauge can be achieved in every case, not all possible choices of effective variables will work.

### 5.3.2. Post-Newtonian Theory

The lowest-order parts of $T^{\mu\nu}{}_{;\nu} = 0$ very obviously give equations that are in the form of the Newtonian equations of motion in post-Newtonian theory, as this is exactly how the Newtonian limit is derived in the context of relativistic gravity. The challenge in this case is therefore to put the equations of motion at first post-Newtonian order into the form of the Newtonian equations.

The relativistic field equations and equations of motion, to the required orders, were derived in Sections 3.3.2 and 3.3.3, respectively. If we consider the time component of $T^{\mu\nu}{}_{;\nu} = 0$ to order $\eta^5$ we see that we can write the equation of relativistic energy conservation in the form of the Newtonian equation of mass conservation (5.9), as long as we have $P = 0$ (i.e. dust). In this case the effective Newtonian variables are as follows:

$$\tilde{\mu} = \mu \left(1 + \frac{1}{2}v^2 - 3U + \Pi + \nabla^2 E\right) + O(\eta^5) \tag{5.37}$$

and

$$\tilde{v}^j = v^j \left(1 - \frac{1}{2}v^2 + U\right) + O(\eta^4)\,, \tag{5.38}$$

where $U$ is the Newtonian potential defined in Eq. (3.92). It is notable that no choice of gauge is yet required in order to put the relativistic energy conservation equation into the form of Eq. (5.9), and that the variables $\tilde{\mu}$ and $\tilde{v}^i$ therefore exist in all possible gauges.

The space component of $T^{\mu\nu}{}_{;\nu} = 0$ to order $\eta^6$ is more complicated, but we find that it can be written in the form in Eq. (5.10) if the following is true:

$$\begin{aligned} 0 \;\; = \;\; & -3v^j U' - \mathcal{H} v^j v^2 + v^2 U_{,j} - 4 v^j v^k U_{,k} + 2 U U_{,j} \\ & -2E_{,ij} U_{,i} + 2 v^k E'_{,jk} + v^k v^n E_{,jkn} - 2F_{(i,j)} U_{,i} \\ & +2 v^k F'_{(j,k)} + v^k v^n F^j{}_{,nk} + \phi^{(4)}_{,j} - (\tilde{U} - U)_{,j} \\ & +B'_{,j} + \mathcal{H} B_{,j} - S'_j - \mathcal{H} S_j - 2 v^k S_{[j,k]}\,, \end{aligned}$$

where we have divided through by a common factor of $\mu$ so that this equation is of order $\eta^4$, and where it has been assumed that $h_{ij} = 0 = P$. The expression above represents three separate equations, with four degrees of freedom in the choice of gauge. It is expected that all of these equations should be able to be satisfied in many ways (probably infinitely many ways), with one degree of gauge freedom remaining.

Manipulating the above expression, using the solutions to the field equations given in Section 3.3.2, as well as the identities in Section 3.3.3, allows us to write this as the following differential equation:

$$\boxed{\frac{d^2 \mathit{\Gamma}_j}{d\tau^2} + \mathcal{H} \frac{d\mathit{\Gamma}_j}{d\tau} + U_{,ij} \mathit{\Gamma}_j - \Phi_{7i,ij} = f}\,, \tag{5.39}$$

where we have defined $\mathit{\Gamma}_j = E_{,j} + F_j$, and where we have introduced the material derivative

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau} + v^i \frac{\partial}{\partial x^i} \tag{5.40}$$

and the potential $\Phi_{7i}$, which is defined by

$$\Phi_{7i} = -\,a^2(\tau) \int \frac{\hat{\mu}\hat{\Gamma}_i'}{|\boldsymbol{x} - \hat{\boldsymbol{x}}|}\, d^3\hat{x}\,. \tag{5.41}$$

The source function in Eq. (5.39) is a function of the potentials given in Section 3.3.2, such that $f = f(U, v^i, V^i, \Phi_1, \Phi_2, \delta\Phi_2, \mathcal{A}, \mathcal{B})$, and is given explicitly by

$$\begin{aligned} f = &-2\Phi_{1,j} - 6\Phi_{2,j} + 5\delta\Phi_{2,j} + \frac{1}{2}\mathcal{A}_{,j} + \frac{1}{2}\mathcal{B}_{,j} \\ &- 4V_j' - 4\mathcal{H}V_j - 8v^i V_{[j,i]} - 3v^j V_{i,i} \\ &- 2\left(U^2\right)_{,j} - 3\mathcal{H}v^j U + \mathcal{H}v^j v^2 - v^2 U_{,j} + 4v^j v^i U_{,i}\,. \end{aligned} \tag{5.42}$$

All of the potentials in this expression can be determined from post-processing Newtonian N-body simulations, and in writing $f$ in this way we have chosen to eliminate the vector gravitational potential $W_i$ using the identities in Section 3.3.2.

Putting the metric into Newtonian motion gauge to first post-Newtonian order requires choosing a gauge such that Eq. (5.39) is true. Solving this equation will almost certainly have to be done numerically, but once numerical solutions have been obtained then it is clear from Section 3.2.4 that the gauge can be fixed by a suitable choice of $\xi^i$, as can be seen from Eqs. (3.66). This leaves total gauge freedom in the time component of $\xi^0$, which can be set to any convenient value whilst still maintaining the required property that the equations of motion of test particles obey equations of the same form as they do in Newtonian physics. This is consistent with the idea that the Newtonian motion gauge is a choice of spatial coordinates, not a choice of temporal gauge.

Once in this gauge, all relativistic gravitational degrees of freedom can be derived by inverting Eqs. (5.37) and (5.38), and then by using the solutions given in Section 3.3.2 for the metric perturbations, together with the numerical solutions for $E$ and $F_i$, which can be obtained from $\Gamma_i$. This gives enough information to calculate *all* relativistic gravitational effects up to first post-Newtonian order, by post-processing a Newtonian N-body simulation. It is remarkable that this is possible, and that one can in principle obtain a relativistic simulation in this way. We have made no approximations in obtaining this result other than the fluid being dust, which includes the particle interpretation by simply taking the mass density to be $\mu(\boldsymbol{x}) = \sum_i m_i \delta(\boldsymbol{x} - \boldsymbol{x}_i)$, for $i$ particles with masses $m_i$ and positions $\boldsymbol{x}_i$.

## 5.4. Discussion

We have considered the structure of gauge transformations in both cosmological perturbation theory (applicable on large scales) and post-Newtonian perturbation theory (applicable on small scales). While both treatments of gravitational fields have their own well defined gauge problems, we find that most of the particular gauge choices that are used in cosmology are not valid using post-Newtonian theory in the presence of non-linear structures. In particular, the *spatially flat* gauge, the *synchronous* gauge, the *comoving orthogonal* gauge, the *total matter* gauge, the *N-body* gauge, and the *uniform density* gauge are all beyond the limits of what it is possible to achieve by applying an infinitesimal coordinate transformation in the post-Newtonian sector.

In contrast, the *longitudinal* gauge and the *Newtonian motion* gauge both appear to be well-defined in the post-Newtonian treatment of gravitational fields, as well as in cosmological perturbation theory. The former is a very simple choice of gauge, and is already well-known to give sensible results when extrapolating the cosmological perturbation theory to the regime of non-linear density contrasts. Here we formalise this result, and explain its veracity, by showing it can be realised in post-Newtonian expansions (which are purposefully constructed to model weak-field gravity in such situations). The latter gauge choice (Newtonian motion gauge) requires numerical integration of a non-local differential equation (5.39) in order to be applied in practise. If this is possible, then it should allow one to post-process cosmological Newtonian N-body simulations in order to derive relativistic corrections to gravitational fields, and to determine the effects of these fields on observables *without* having to perform additional simulations. This is an intriguing possibility, which it would be interesting to apply in practice.

Our results provide support for the use of longitudinal gauge in studies that attempt to simultaneously model both small-scale non-linear structures as well as linear structures on large scales, see e.g. the numerical code *gevolution* [122] or the two-parameter perturbative approach that we will come to discuss subsequently. On the other hand, they provide a warning that other choices of gauge should be applied with care. In particular, the fact that one cannot use gauge transformations to realise synchronous coordinates in post-Newtonian theory has potentially interesting consequences. While this result does *not* imply that it is impossible in general to find a coordinate system where the time coordinate corresponds to the proper time of observers comoving with matter[1], it does means that the coordinates of a synchronous coordinate system and the coordinates of a post-Newtonian perturbed FLRW space-time cannot be related by a small gauge generator. That is, the difference between these two different notions of time is large, in the sense defined by the perturbative expansion, and is therefore unattainable by small gauge transformations. Such a result would appear to have significance for a number of studies that use proper time in the presence of non-linear structures, such as the calculation of galaxy bias on hypersurfaces of constant proper time [123]. An interesting discussion of the use of a comoving-synchronous coordinate system is given in [124].

It may also go some way to explaining the vastly different expectations that different groups of cosmologists appear to have when considering the problem of cosmological back-reaction (see e.g. [38] and [34]).

[1] In fact, this is always possible for dust [81].

# 6. Two-parameter perturbation theory

In this chapter, we will review the two-parameter perturbation theory framework, first constructed in [125], and further developed in [126] and [127]. The initial concept for the formalism was developed by Sophia Goldberg, Timothy Clifton and Karim Malik. The extension to include a cosmological constant and background radiation was carried out by Sophia Goldberg. I restructured formalism in terms of effective fluid quantities and combined variables. The derivation of the stress-energy conservation equations and confirmation of consistent time evolution was carried out by myself in [127].

## 6.1. Motivation

Developing a mathematical formalism capable of handling the nonlinear properties of gravity in cosmology is a highly non-trivial task. There is now a substantial literature dedicated to developing different approaches to modelling nonlinear gravitational physics in cosmology. The most common approach is a direct implementation of second-order cosmological perturbation theory [128–130], which allows relativistic gravitational perturbations around a homogeneous and isotropic background to be modelled in the presence of small density contrasts. Other approaches, however, have started to import techniques from post-Newtonian gravity [42, 52, 69], where gravitational fields are assumed to be slowly varying and where nonlinear density contrasts can be consistently modelled.

The standard method for dealing with gravity in cosmology has been to use cosmological perturbation theory to model linear structures on scales above the homogeneity scale ($\gtrsim$ 100Mpc), and Newtonian gravity to model nonlinear structures on smaller scales. The two different approaches detailed above are the obvious nonlinear extensions of this methodology. This looks very natural at linear order in the gravitational fields, partly because the linearised equations of Newtonian gravity can be recovered as the quasi-static limit of cosmological perturbation theory (when the gravitational fields slowly vary in time). Indeed, this pleasing feature extends to nonlinear gravitational fields in cosmology [52].

However, if one wants to consider nonlinear gravity in a universe that *simultaneously* contains linear structures on large scales *and* nonlinear structures on small scales, then one must adopt a more sophisticated approach. This is because, when considering the quasi-static limit, the terms that have might have been relegated to higher-order in perturbation theory can no longer be entirely forgotten; they can and should be expected to appear in the leading-order gravitational field equations. This could be at second-order on small scales, but could in principle be at what is usually thought of as first-order on large scales.

We present a formalism for realistic cosmological models that contain relativistic fluids with barotropic equations of state, as well as a cosmological constant, $\Lambda$, and non-relativistic dust-like matter that can be used to model dark matter and baryons. The result is a set of equations that can be used to calculate the effect of small-scale structure on the leading-order perturbations on large scales. These equations contain terms that are quadratic in short-scale potentials and can be written as an effective fluid [41], as well as terms that couple scalar, vector and tensor perturbations in the large-scale cosmology. Both of these two types of terms offer exciting possibilities for testing nonlinear gravity with upcoming surveys. Much work remains to be done however; for example including a consistent two-parameter bias prescription and testing the formalism with simulations is vital for application to galaxy number count surveys, whilst modelling of the intrinsic alignment issue is required for application to lensing surveys [112, 131].

## 6.2. Perturbative framework

Both post-Newtonian and cosmological perturbations should be expected to exist in any realistic model of the Universe [40]. In order to describe a universe with slowly-changing nonlinear density contrasts on small-scales and linear fluctuations on large scales, we will employ a post-Newtonian expansion and cosmological perturbation theory simultaneously. For the former of these expansions we use the expansion parameter $\eta \ll 1$, while for the latter we use $\epsilon \ll 1$. We assume that any field $Q$ can be expanded in both $\epsilon$ and $\eta$ as follows:

$$Q = \sum_{n,m} \frac{1}{n!m!} Q^{(n,m)} \,, \tag{6.1}$$

where $Q^{(n,m)}$ is a quantity of order $\mathcal{O}(\epsilon^n \eta^m)$. The characteristic length scales on which post-Newtonian and cosmological perturbations exist and vary on will be labelled $L_N$ and $L_C$, respectively.

### 6.2.1. Two-parameter perturbation theory

Our two-parameter expansion in both $\epsilon$ and $\eta$ will be constructed around a flat FLRW geometry, corresponding to the line-element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left\{ dr^2 + r^2 d\,\Omega^2 \right\} . \tag{6.2}$$

This is the standard background for cosmological perturbation theory, but has been used less commonly in post-Newtonian gravity (see however [42, 52]). Nevertheless, it has been shown that both expansions can be performed in such a background in an entirely self-consistent and well-posed way [53, 83, 125]. We will use coordinate time here as opposed to conformal time, since the formal development of the two-parameter expansion was carried out in coordinate time. We will subsequently switch to conformal time when it comes to discussion of approximate solutions, since conformal time makes many expressions slightly more compact. We refer the reader to Chapter 3 for a development of a post-Newtonian expansion on an FLRW background.

The first step in doing this is to expand the total energy density and pressure in both $\epsilon$ and $\eta$:

$$\rho = \rho^{(0,0)} + \rho^{(0,2)} + \rho^{(1,0)} + \rho^{(1,1)} + \rho^{(1,2)} + \tfrac{1}{2}\rho^{(0,4)} + \dots \tag{6.3}$$

$$P = P^{(0,0)} + P^{(1,0)} + P^{(1,2)} + \tfrac{1}{2}P^{(0,4)} + \dots . \tag{6.4}$$

The terms $\rho^{(0,0)}$ and $P^{(0,0)}$ can be considered as the background energy density and pressure, as they are not perturbed in either $\epsilon$ or $\eta$. All other terms correspond to perturbations at the order indicated by the superscript, but we have neglected to include $\delta$ symbols before them to keep the notation as compact as possible. To be even more precise, the orders of magnitude of these perturbed quantities are given by

$$\begin{aligned} \rho^{(0,0)} &\sim \frac{1}{L_C^2}\,, & \rho^{(n,0)} &\sim \frac{\epsilon^n}{L_C^2}\,, \\ \rho^{(0,m)} &\sim \frac{\eta^m}{L_N^2} \quad \text{and} & \rho^{(n,m)} &\sim \frac{\epsilon^n \eta^m}{L_N^2}\,, \end{aligned} \tag{6.5}$$

where $\{m, n\} \in \mathbb{N}^+$, and again $L_C$ and $L_N$ are the characteristic length scales of the cosmological and post-Newtonian sytems, respectively. A similar expression holds for the expansion of $P$. The length scales are necessary in the denominators of these expressions, as $\rho$ is a quantity with dimension $L^{-2}$, and because it only makes sense to compare the magnitude of quantities with the same dimensions. The first thing to notice about Eq. (6.3) is that the mixed-order terms do not always appear at the same order as the product of post-Newtonian and cosmological terms (i.e. we have included $\rho^{(1,1)}$, even though there is no $\mathcal{O}(\eta)$ term in the post-Newtonian expansion). The reason for this is that such terms are necessarily generated by arbitrary gauge transformations, and so cannot be assumed to vanish in general, even if they are assumed to do so in one particular coordinate system.

We also need to expand the metric in both $\epsilon$ and $\eta$, which we do as follows:

$$\begin{aligned} g_{00} &= g_{00}^{(0,0)} + g_{00}^{(0,2)} + g_{00}^{(1,0)} + g_{00}^{(1,1)} + g_{00}^{(1,2)} + \tfrac{1}{2} g_{00}^{(0,4)} + \dots \\ &= -1 + h_{00}^{(0,2)} + h_{00}^{(1,0)} + h_{00}^{(1,1)} + h_{00}^{(1,2)} + \tfrac{1}{2} h_{00}^{(0,4)} + \dots \end{aligned} \tag{6.6}$$

$$\begin{aligned} g_{ij} &= g_{ij}^{(0,0)} + g_{ij}^{(0,2)} + g_{ij}^{(1,0)} + g_{ij}^{(1,1)} + g_{ij}^{(1,2)} + \tfrac{1}{2} g_{ij}^{(0,4)} + \dots \\ &= a^2 \left( \delta_{ij} + h_{ij}^{(0,2)} + h_{ij}^{(1,0)} + h_{ij}^{(1,1)} + h_{ij}^{(1,2)} + \tfrac{1}{2} h_{ij}^{(0,4)} \right) + \dots \end{aligned} \tag{6.7}$$

$$\begin{aligned} g_{0i} &= g_{0i}^{(1,0)} + g_{0i}^{(0,3)} + g_{0i}^{(1,2)} + \dots \\ &= a \left( h_{0i}^{(1,0)} + h_{0i}^{(0,3)} + h_{0i}^{(1,2)} \right) + \dots \, , \end{aligned} \tag{6.8}$$

where in the second line of each of these equations we have chosen our background metric $g_{\mu\nu}^{(0,0)}$ to be the flat FLRW metric from Eq. (6.2), and simultaneously defined the perturbations $h_{\mu\nu}$. The orders of magnitude of each of the perturbations to each of the components of this metric are the minimal set required to self-consistently account for the gravitational fields of the two-parameter perturbed perfect fluid discussed above in any arbitrary coordinate system. We find that the inclusion of radiation and a cosmological constant does not require the introduction of any new metric potentials at any new order.

The final ingredient of the field equations that must be perturbed is the peculiar velocity, $v^i$. This is split into post-Newtonian and cosmological parts such that

$$v^i = v^{(0,1)i} + v^{(1,0)i} + \dots \, , \tag{6.9}$$

which leads to the following components of the reference four-velocity $u^\mu$:

$$u^0 = 1 + \frac{1}{2}\left(h^{(0,2)}_{00} + h^{(1,0)}_{00}\right) + \tfrac{1}{2}v^{(0,1)i}v^{(0,1)}_i + \dots \tag{6.10}$$

$$u^i = \frac{1}{a}\left(v^{(0,1)i} + v^{(1,0)i}\right) + \dots\,, \tag{6.11}$$

which are derived using the normalization condition $u^\mu u_\mu = -1$, and Eqs. (6.6)-(6.8). The components of the two-parameter perturbed energy-momentum tensor that arise from these equations are given in Appendix (A). The components of the Ricci tensor in this scenario were calculated in Ref. [125], we repeat them in Appendix (A) for the reader's convenience.

The reader should note that within the context of the two-parameter formalism, time derivatives are taken to add an extra order-of-smallness, $\eta$, compared to spatial derivatives whenever they act on an object that contains any non-zero perturbation in its post-Newtonian sector. So, for example, we take

$$\dot{\rho}^{(0,2)} \sim \eta\,|\nabla\rho^{(0,2)}| \sim \frac{\eta^3}{L_N^3} \tag{6.12}$$

$$\dot{\rho}^{(1,1)} \sim \eta\,|\nabla\rho^{(1,1)}| \sim \frac{\epsilon\eta^2}{L_N^3} \tag{6.13}$$

$$\dot{\rho}^{(1,0)} \sim |\nabla\rho^{(1,0)}| \sim \frac{\epsilon}{L_C^3}\,, \tag{6.14}$$

where dots indicate derivatives taken with respect to *coordinate time* as opposed to conformal time. As in Eq. (6.5), the purpose of this is to reflect the expectation that quantities perturbed in the post-Newtonian sector should be slowly varying in time and change over spatial length scales $L_N$, while quantities that are perturbed only in the cosmological sector should vary equally over both time and length scales $L_C$.

### 6.2.2. Including radiation and Λ

Let us now consider how to add radiation and $\Lambda$ to our two-parameter expansion. For radiation this can be achieved by writing

$$\rho = \rho_M + \rho_R \tag{6.15}$$

$$P = P_M + P_R \tag{6.16}$$

where $\rho_M$ and $P_M$ are the energy density and pressure of non-relativistic matter, $\rho_R$ and $P_R$ are the energy density and pressure of radiation, and where we take $v^i_M$ and $v^i_R$ to be the peculiar velocities of the matter and radiation fluids. We then want to expand each of these new quantities in $\epsilon$ and $\eta$, which we do according to

$$\rho_M = \rho_M^{(0,2)} + \rho_M^{(1,0)} + \rho_M^{(1,1)} + \rho_M^{(1,2)} + \tfrac{1}{2}\rho_M^{(0,4)} + \dots \tag{6.17}$$
$$P_M = P_M^{(1,0)} + P_M^{(1,2)} + \tfrac{1}{2}P_M^{(0,4)} + \dots \tag{6.18}$$
$$\rho_R = \rho_R^{(0,0)} + \rho_R^{(1,0)} + \rho_R^{(1,2)} + \tfrac{1}{2}\rho_R^{(0,4)} + \dots \tag{6.19}$$
$$P_R = P_R^{(0,0)} + P_R^{(1,0)} + P_R^{(1,2)} + \tfrac{1}{2}P_R^{(0,4)} + \dots\,, \tag{6.20}$$

and

$$v^i_M = v_M^{(0,1)i} + v_M^{(1,0)i} + \dots\,, \qquad v^i_R = v_R^{(0,1)i} + v_R^{(1,0)i} + \dots\,. \tag{6.21}$$

These equations can, of course, be compared to Eqs. (6.3), (6.4) and (6.9) to read off values for the perturbations to the total energy density, pressure and $v^i$.

The reader will note that the expansions of the matter and radiation fluids have not been performed in an identical way: We have omitted (i) a time-dependent background-level contribution to the matter energy density and pressure, and (ii) a Newtonian-level contribution to the radiation energy density and pressure, so that

$$\rho_M^{(0,0)} = P_M^{(0,0)} = 0 \qquad \text{and} \qquad \rho_R^{(0,2)} = P_R^{(0,2)} = 0\,.$$

The $\rho_M^{(0,0)}$ term is neglected because it corresponds to a term that could otherwise be as large as the Newtonian rest mass energy density $\rho_M^{(0,2)}$, which we consider to be highly unphysical.

In the real universe there is no time-dependent background matter component to the energy density, $\rho_M^{(0,0)}(t)$. This is because the leading-order contribution to $\rho_M$ is in fact dominated by the (inhomogeneous) rest mass of galaxies, dust *etc.*, which is exactly what $\rho_M^{(0,2)}(x^\mu)$ corresponds to. Furthermore, $\rho_M^{(0,0)}$ would necessarily have to be a function of time only and because there is no discernible homogeneous fluid of non-relativistic matter with this magnitude in the real Universe. The term $P_M^{(0,0)}$ could be neglected on similar grounds, but must also vanish because of the requirement $P \ll \rho$ in non-relativistic matter.

Let us now consider the expansion of $\rho_R$ and $P_R$ given in Eqs. (6.19) and (6.20). For this purpose it is useful to consider the stress-energy conservation equation for the total stress-energy tensor $T_{\mu\nu}$:

$$\nabla^{\mu} T_{\mu\nu} = \nabla^{\mu}(T_{M\mu\nu} + T_{R\mu\nu}) = 0\,, \tag{6.22}$$

where $T_{M\mu\nu}$ and $T_{R\mu\nu}$ are the matter and radiation contributions to the total stress-energy tensor, respectively. This implies $\nabla^{\mu} T_{M\mu\nu} = Q_{\nu}$ and $\nabla^{\mu} T_{R\mu\nu} = -Q_{\nu}$, where $Q_{\nu} \neq 0$ for interacting fluids and $Q_{\nu} = 0$ for non-interacting fluids. In either case, the lowest-order part of Eq. (6.22) is given by

$$\nabla P_R^{(0,0)} = 0\,, \tag{6.23}$$

which implies $P_R^{(0,0)} = P_R^{(0,0)}(t)$ is a function of time only. If we now take $P_R = \frac{1}{3}\rho_R$, then this result implies that the leading-order part of the energy density in radiation must also be spatially homogeneous, such that $\rho_R^{(0,0)} = \rho_R^{(0,0)}(t)$. This is, in fact, exactly what is required for a background-level contribution to the energy density in an FLRW model.

A similar argument can now be used to understand why it would be inappropriate to include a term $\rho_R^{(0,2)}$ in Eq. (6.19). Such a term would imply the existence of $P_R^{(0,2)}$ which, again through the conservation equations, can be shown to be necessarily spatially homogeneous. Such a term would therefore be functionally degenerate with $\rho_R^{(0,0)}$, as they are both functions of time only, and would therefore show up in every conceivable set of equations in exactly the same way. We can therefore neglect both $\rho_R^{(0,2)}$ and $P_R^{(0,2)}$ without any loss of generality. Moreover, the term $\rho_R^{(0,2)}(t)$ would be Newtonian in size, and such a term would be highly unusual in normal post-Newtonian gravity. We therefore find that the lowest order at which inhomogeneous perturbations in radiation fit into our two-parameter expansion is at order $\mathcal{O}(P_R^{(1,0)}) \sim \mathcal{O}(\epsilon L_C^{-2})$, which corresponds to a cosmological-scale perturbation.

The reader may also note that there is no term $\rho_R^{(1,1)}$ in Eq. (6.19), whereas there is a term $\rho_M^{(1,1)}$ in Eq. (6.17). The $\rho_M^{(1,1)}$ is necessary because a term of the form $\rho_{M,i}^{(0,2)}\xi^{(1,0)i}$ is always generated under a general infinitesimal gauge transformation [125] (where $\xi^{(1,0)i}$ is a part of the gauge generator – see Section 6.3). This implies there must in general exist a term $\rho_M^{(1,1)}$ in the expansion of $\rho_M$, because even if we artificially exclude it in one coordinate system, it will be generated in another. However, a similar argument does not apply to $\rho_R^{(1,1)}$, because the gauge transformation $\rho_R^{(0,0)}$ does not generate any terms of the same order as $\rho_R^{(1,1)}$. This can be seen to be true because $\rho_R^{(0,0)}$ is a function of time only, such that $\rho_{R,i}^{(0,0)}\xi^{(1,0)i} = 0$. Of course, the same argument would apply to a term of the form $\rho_R^{(0,2)}$, if it had been included, as this term is also time dependent. This means that we can set $\rho_R^{(1,1)} = P_R^{(1,1)} = 0$ in any coordinate system, and the same result will hold in any other coordinate system related by an infinitesimal gauge transformation.

Finally, let us consider the cosmological constant $\Lambda$. We assign an order of magnitude and dimensions to the cosmological constant in the following way:

$$\Lambda = \Lambda^{(0,0)} \sim \frac{1}{L_C^2} \,. \tag{6.24}$$

This choice is motivated by the fact that the cosmological constant in the standard model of cosmology must be of background order, in order for it to be influential in the Friedmann equations at late times. There is also no point in perturbing it in either $\epsilon$ or $\eta$, as it is a constant, and the Taylor expansion is trivial. The cosmological constant therefore fits naturally into our two-parameter expansion at lowest-order, as a cosmological background quantity with corresponding scale $L_C^{-2}$.

Having expanded the relevant quantities in our formalism, we must proceed to discuss the two-parameter version of the gauge problem.

## 6.3. Constructing gauge-invariant variables

As discussed before, a general gauge transformation between coordinate systems can be written as

$$x^\mu \mapsto \tilde{x}^\mu = e^{\xi^\alpha \partial_\alpha} x^\mu \,, \tag{6.25}$$

where $\xi^\mu$ is the gauge generator. All tensors, $\mathcal{T}$, are taken to transform under the gauge transformation in Eq. (6.25) as

$$\tilde{\mathcal{T}} = e^{\mathcal{L}_\xi} \mathcal{T} = \mathcal{T} + \mathcal{L}_\xi \mathcal{T} + \tfrac{1}{2} \mathcal{L}_\xi^2 \mathcal{T} + \dots \,, \tag{6.26}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative operator with respect to $\xi^\mu$. This exponential map results in an invertible transformation, and can be applied to both the metric and the stress-energy tensor. We must now expand the components of the gauge generator in terms of $\epsilon$ and $\eta$, which we do as follows:

$$\xi^0 = \xi^{(1,0)0} + \xi^{(0,3)0} + \xi^{(1,2)0} + \dots \sim \epsilon L_C + \eta^3 L_N + \epsilon\eta^3 L_N + \dots \tag{6.27}$$

$$\xi^i = \xi^{(1,0)i} + \xi^{(0,2)i} + \xi^{(1,1)i} + \xi^{(1,2)i} + \tfrac{1}{2}\xi^{(0,4)i} \dots$$
$$\sim \epsilon L_C + \eta^2 L_N + \epsilon\eta^2 L_N + \eta^4 L_N \dots . \tag{6.28}$$

These non-vanishing components of the gauge generator have been chosen so that no new components of the metric or the stress-energy tensor are generated by this transformation, which is an important condition to ensure the problem is being treated in a self-consistent manner. Eq. (6.26) was used in [125] and [126] to explicitly construct a set of two-parameter gauge invariant variables. We present the relevant results from those papers in Appendix B.

## 6.4. Constructing the field equations

The two-parameter expansion described in the previous sections could in principle be applied to numerous different physical systems. While the perturbed metric and stress-energy tensor can be written down without specifying any specific relationship between either $\epsilon$ and $\eta$ or $L_C$ and $L_N$, we must choose how to express these quantities in terms of one another if we want to be able to solve a hierarchical set of field equations. In order to model a realistic universe that has nonlinear structure on scales up to $\sim$ 100Mpc, as well as linear structure on large scales, we choose $L_N/L_C \sim \eta$. On the other hand, to model a realistic universe, gravitational potentials must have similar magnitude on both small and large scales, so we choose $\epsilon \sim \eta^2$. Both of these requirements are therefore satisfied by the choice

$$\epsilon \sim \eta^2 \sim \frac{L_N^2}{L_C^2} \sim 10^{-5} \,, \tag{6.29}$$

where $10^{-5}$ is the typical depth of a potential on both cosmological and post-Newtonian scales. With these relations we can translate our two-parameter expansion into effectively a single-parameter expansion in $\eta$, and write the field equations order-by-order in $\eta$ in terms of the gauge-invariant variables defined in Appendix B. We further choose to express the field equations in units of $L_N^{-2}$. This last choice has no particular physical significance, and is purely for expediency. The full set of two-parameter field equations, written in terms of gauge invariant variables are given in Appendix C.

The only quantity we have not considered the sizes of derivatives of so far is the scale factor, $a(t)$. Since $a(t)$ is expected to change on cosmological timescales, we should expect

$$\dot{a} \sim \frac{1}{L_C} \sim \eta \frac{1}{L_N} \,. \tag{6.30}$$

Since we want to write the field equations in units of $L_N^{-2}$, it then follows that we must also remember to add additional factors of $\eta$ when taking time derivatives of the scale factor.

In order to assist the reader in working out the sizes of the various derivatives that appear in the field equations, we provide a short collection of "rules of thumb".

(i) Taking spatial derivatives of cosmologically perturbed quantities adds a factor of $\frac{\eta}{L_N}$.

(ii) Taking spatial derivatives of post-Newtonian or mixed quantities adds a factor of $\frac{1}{L_N}$.

(iii) Taking time derivatives adds factors of $\frac{\eta}{L_N}$ to all quantities.

Given these rules, one can consistently assign orders of magnitudes and units to all quantities and derivatives appearing in the gravitational field equations and stress-energy conservation equations, and thus write down a consistent hierarchy of perturbation equations that can be solved order by order.

## 6.5. Perturbed Field Equations

### 6.5.1. Notation

The equations presented in Appendix C.2 constitute a hierarchy of field equations written in gauge-invariant variables. Equations (C.16) and (C.17) are the leading order results. Unfortunately, the notational scheme that was used to derive these is cumbersome, and ignores the fact that some degrees of freedom should not be regarded as truly independent. We will tidy up the notation here, so that the rest of the thesis may be easier for the reader to understand, and also so as to bring the notation more into line with that traditionally used in the rest of the literature, facilitating the comparison of results.

At present, Eqs. (C.18)-(C.27) from Appendix C.2 contain a total of sixteen degrees of freedom: six scalars ($\Phi^{(1,0)}$, $\Phi^{(1,2)}$, $\Phi^{(0,4)}$, $\Psi^{(1,0)}$, $\Psi^{(1,2)}$ and $\Psi^{(0,4)}$), six in the tensors ($\mathbf{h}_{ij}^{(1,0)}$, $\mathbf{h}_{ij}^{(1,2)}$ and $\mathbf{h}_{ij}^{(0,4)}$) and four in the vectors ($\mathbf{B}^{(1,0)}$ and $\mathbf{B}^{(1,2)}$). Taking into account the four degrees of freedom removed by gauge fixing, we still have an excess of six degrees of freedom, given that there can only be 10 degrees of freedom in the metric. The implication is that six of these degrees of freedom are illusory, resulting merely from the separation of scales that has been employed. The removal of these degrees of freedom is achieved by defining new sets of composite variables as follows:

$$\mathrm{U} \equiv -\tfrac{1}{2}\left(\Phi^{(0,2)} + \Phi^{(1,1)}\right) \tag{6.31}$$
$$\phi \equiv -\tfrac{1}{2}\left(\Phi^{(1,0)} + \Phi^{(1,2)} + \tfrac{1}{2}\Phi^{(0,4)}\right) \tag{6.32}$$
$$\psi \equiv \tfrac{1}{2}\left(\Psi^{(1,0)} + \Psi^{(1,2)} + \tfrac{1}{2}\Psi^{(0,4)}\right) \tag{6.33}$$
$$\mathrm{S}_j \equiv -\left(\mathbf{B}_j^{(1,0)} + \mathbf{B}_j^{(0,3)} + \mathbf{B}_j^{(1,2)}\right) \tag{6.34}$$
$$\mathrm{h}_{ij} \equiv \tfrac{1}{4}\left(\mathbf{h}_{ij}^{(1,0)} + \mathbf{h}_{ij}^{(1,2)} + \tfrac{1}{2}\mathbf{h}_{ij}^{(0,4)}\right)\ . \tag{6.35}$$

These quantities all represent fluctuations about a spatially-flat FLRW geometry, which in a particular choice of coordinates can be written as

$$ds^2 = a^2(\tau)\Big[-(1+2\mathrm{U}+2\phi)d\tau^2+\big((1-2\mathrm{U}-2\psi)\delta_{ij}+2\mathrm{h}_{ij}\big)dx^i dx^j - 2\mathrm{S}_i d\tau dx^i\Big]\, , \tag{6.36}$$

where $a$ is the scale factor and $\tau$ is a conformal time coordinate. We can do the same thing for the matter variables:

$$\bar{\rho} \equiv \bar{\boldsymbol{\rho}}_R^{(0,0)} + \bar{\boldsymbol{\rho}}_M^{(0,2)} \tag{6.37}$$
$$\bar{\mathrm{p}} \equiv \mathbf{P}^{(0,0)} \tag{6.38}$$
$$\delta\rho_{\mathrm{N}} \equiv \delta\boldsymbol{\rho}^{(0,2)} + \boldsymbol{\rho}^{(1,1)} \tag{6.39}$$
$$\delta\rho \equiv \boldsymbol{\rho}^{(1,0)} + \boldsymbol{\rho}^{(1,2)} + \tfrac{1}{2}\boldsymbol{\rho}^{(0,4)} \tag{6.40}$$
$$\delta \mathrm{p} \equiv \mathbf{P}^{(1,0)} + \mathbf{P}^{(1,2)} + \tfrac{1}{2}\mathbf{P}^{(0,4)} \tag{6.41}$$
$$\mathrm{v}_{\mathrm{N}i} \equiv \mathbf{v}_i^{(0,1)} \tag{6.42}$$
$$\mathrm{v}_i \equiv \mathbf{v}_i^{(1,0)}\, . \tag{6.43}$$

These matter variables can be considered to be part of the following stress-energy tensor:

$$T_{\mu\nu} = (\bar{\rho} + \delta\rho_{\mathrm{N}} + \delta\rho + \bar{p} + \delta p)u_\mu u_\nu + (\bar{p} + \delta p)g_{\mu\nu} \,. \tag{6.44}$$

A number of these new variables could be considered to be "composite quantities", as they contain a number of different perturbative orders in the same variable. For example, the variable $\psi$ is dominated by $\mathcal{O}(\epsilon)$ terms on cosmological length scales $L_C$, but contains smaller terms at $\mathcal{O}(\eta^4)$ on small-scales $L_N$. This is quite atypical in perturbation theory. However, the way in which these quantities arise together in the field equations suggest that they should be solved for together.

## 6.5.2. Gravitational field equations

In this section we will present the perturbed field equations that result from simultaneously considering linear structures on large scales, and nonlinear single-stream structures on small scales.

### Leading order

After simultaneously expanding the field equations in post-Newtonian and cosmological perturbation theories we find the leading-order parts are given by the effective Friedmann equations (C.16) and (C.17), which written in terms of the variables given in Section 6.5.1 are equivalent to

$$\mathcal{H}^2 = \frac{8\pi a^2}{3}\bar{\rho} + \frac{1}{3}\Lambda a^2 + \mathcal{O}(\eta^4 L_N^{-2}) \,, \tag{6.45}$$

and

$$\mathcal{H}' = -\frac{4\pi a^2}{3}(\bar{\rho} + 3\bar{p}) + \frac{1}{3}\Lambda a^2 + \mathcal{O}(\eta^4 L_N^{-2}) \tag{6.46}$$

These equations have been separated out from the Newton-Poisson equation that governs the inhomogeneous matter component, $\delta\rho_{\mathrm{N}}$, via averaging in exactly the same fashion as was done in Chapter 3. The Newtonian gravitational field equation, which occurs at the same order in our expansion, is given by

$$\nabla^2 \mathrm{U} = 4\pi a^2 \delta\rho_{\mathrm{N}} + \mathcal{O}(\eta^4) \,, \tag{6.47}$$

where $\eta$ is the expansion parameter for the post-Newtonian expansion. In this case it is used to characterise the size of structure on scales of order the homogeneity scale;

the largest-scale at which the post-Newtonian expansion can sensibly be performed. The reader may note that only dark matter and baryonic matter contribute to $\delta\rho_{\mathrm{N}}$, and not radiation.

These equations take precisely the same functional form as Equation (3.88) and Equation (3.89) from Chapter 3, apart from the fact that they have been generalised such that they could include a background radiation component. However, Equation (6.47) contains a hidden subtlety; whilst here we have combined Poisson equations for $\Phi^{(0,2)} + \Phi^{(1,1)}$ to obtain a Newton-Poisson equation valid up to $\mathcal{O}(\eta^4 L_N^{-2})$, the solutions for $\Phi^{(0,2)}$ and $\Phi^{(1,1)}$ will not be of the same form, since the system must be closed by the stress-energy conservation equations (as in regular Newtonian gravity). In particular, $\boldsymbol{\rho}^{(1,1)}$ satisfies a modified conservation equation, with additional source terms arising from couplings with the cosmological sector of the expansion, and so its time evolution will be different to that of $\boldsymbol{\rho}^{(0,2)}$. We will discuss this issue and its interpretation in greater detail in Section 6.6.

### Subleading order

This order corresponds to what would traditionally be considered *first-order* in cosmological perturbations, but *second-order* in short-scale Newtonian gravitational fields. These equations also naturally contain the first exclusively post-Newtonian corrections, $\Phi^{(0,4)}$, and a mixed correction $\Phi^{(1,2)}$ that is functionally degenerate (i.e appears in all the same field equations in all the same ways) with $\Phi^{(0,4)}$, since these terms are contained within the definition of the variable $\phi$. The same logic extends to $\psi$ and $\delta\rho$ *etc.* When considering a product between terms, the reader should understand that higher order product terms i.e $\Psi^{(0,4)}\boldsymbol{\rho}^{(1,1)} \sim \eta^6 L_N^{-2}$ are implicitly truncated. This means that we implicitly understand $\rho_{\mathrm{N}}\psi \sim \boldsymbol{\rho}^{(0,2)}\Psi^{(1,0)}$, but $\nabla^2\psi \sim \nabla^2\left(\Psi^{(1,0)} + \Psi^{(1,2)} + \frac{1}{2}\Psi^{(0,4)}\right)$.

Using the variables defined in Section 6.5.1, we find that Equations (C.22) and (C.23) can be represented as the following two equations for the scalar part of the gravitational field:

$$\begin{aligned}\frac{1}{3}\nabla^2\phi + \mathcal{H}\phi' + \mathcal{H}\psi' + \psi'' + 2\mathcal{H}'\phi = \frac{4\pi a^2}{3}\left(\delta\rho + \delta\rho_{\mathrm{eff}} + 3\delta\mathrm{p} + 3\delta\mathrm{p}_{\mathrm{eff}}\right)\\ + \frac{2}{3}(\mathcal{D}^{ij}\mathrm{U})\mathrm{h}_{ij} - \frac{8\pi a^2}{3}\delta\rho_{\mathrm{N}}(\psi - \phi) + \mathcal{O}(\eta^5)\end{aligned} \tag{6.48}$$

and

$$\frac{1}{3}\nabla^2\psi - \mathcal{H}\psi' - \mathcal{H}^2\phi = \frac{4\pi a^2}{3}\big(\delta\rho + \delta\rho_{\rm eff}\big) + \frac{1}{3}(\mathcal{D}^{ij}\mathrm{U})\mathrm{h}_{ij} - \frac{16\pi a^2}{3}\delta\rho_N\psi + \mathcal{O}(\eta^5)\,, \tag{6.49}$$

where $D_{ij}\varphi \equiv \varphi_{,(ij)} - \frac{1}{3}\delta_{ij}\nabla^2\varphi$ is the symmetric trace-free second derivative operator on any field $\varphi$, and where perturbations in radiation, and dark and baryonic matter contribute to both $\delta\rho$ and $\delta\mathrm{p}$. This set of equations has a fundamental structure that is similar to that of standard cosmological perturbation theory in Poisson gauge, as exemplified by Equations (4.49)-(4.50). This is to be expected since they are the leading order equations for cosmological perturbations. However there are important differences.

The reader will note that these equations contain *extra terms* when compared to those of standard first-order cosmological perturbation theory. Firstly, there are effective energy density and pressure terms, $\delta\rho_{\rm eff}$ and $\delta\mathrm{p}_{\rm eff}$. These are comprised exclusively of leading order terms, that in principle would have already been solved for, and are solely due to the presence of *nonlinear structures* on small scales. They are given explicitly in Eqs. (6.53) and (6.54), below.

Secondly, in the above equations, the Newtonian potential U couples to $\mathrm{h}_{ij}$ and there are extra terms on the right-hand-side of these equations that are linear in $\phi$ and $\psi$. These interaction terms do not exist in standard cosmological perturbation theory and vanish in the limit in which nonlinear small-scale structures vanish. In general, the interaction terms should be expected to produce coupling between scalar, vector and tensor parts of the gravitational field on cosmological scales *and* coupling between different Fourier modes in Fourier-space. Furthermore, they modify the structure of the linear operator for this set of partial differential equations from one that is only functionally dependent on conformal time, to one that is inseparable in space and time, even containing a *stochastic* spatial dependence. Whilst the equations are still linear, this considerably complicates the task of finding solutions, as the whole system is now coupled.

The remaining parts of the gravitational field are the vector and tensor modes. For the vectors we find that we can write the following single equation to describe $\mathrm{S}_i$, accurate up to order $\mathcal{O}(\eta^5)$:

$$\begin{aligned}\nabla^2\mathrm{S}_i + 4\partial_i\big(\psi' + \mathcal{H}\phi\big) + 16\pi a^2\big(\bar\rho + \bar{\mathrm{p}} + \delta\rho_{\rm N}\big)(\mathrm{v}_i - \mathrm{S}_i)\\ = -16\pi a^2 Q_i^{\rm eff} - 8\pi a^2\delta\rho_{\rm N}\mathrm{S}_i - 2(\partial_j\partial_i\mathrm{U})\mathrm{S}^j + \mathcal{O}(\eta^5)\,.\end{aligned} \tag{6.50}$$

We can take the leading-order part of this equation, at $\mathcal{O}(\eta^3)$, and write it as the following simple Poisson equation

$$\nabla^2 \mathrm{S}_i + 4\partial_i(\mathrm{U}' + \mathcal{H}\mathrm{U}) + 16\pi a^2\big(\bar{\rho} + \bar{p}\big)\mathrm{v}_{\mathrm{N}i} \;\; = -16\pi a^2 \delta\rho_{\mathrm{N}}\mathrm{v}_{\mathrm{N}i} + \mathcal{O}(\eta^4)\,. \quad (6.51)$$

The leading-order part of the vector gravitational field, given by the solution to Eq. (6.51), is only sourced by small-scale quantities, and is a hundred times greater than might naively be expected from cosmological perturbation theory. This is the equation that was identified in the post-Friedmann approach of [52], and solved for numerically in [132]. For the full vector equation (6.50), accurate up to $\mathcal{O}(\eta^5)$, it can be seen that there exist sources on both small and large scales and mode-mixing (which is missing from [44] and [43]). This equation has an effective energy flux, $Q_i^{\mathrm{eff}}$, which is due to small scale potentials. It also has extra interaction terms on the right-hand-side that are linear in $S_i$. Both of these vanish when small-scale structures are absent. The explicit expression for $Q_i^{\mathrm{eff}}$ is given in Eq. (6.55), below, along with the other effective fluid quantities.

The final field equation we require, in order to complete our set to the desired accuracy, is given as follows:

$$\begin{aligned}&\nabla^2 \mathrm{h}_{ij} - \mathrm{h}''_{ij} - 2\mathcal{H}\mathrm{h}'_{ij} + \mathcal{D}_{ij}(\phi - \psi) - 2\mathcal{H}\partial_{(j}\mathrm{S}_{i)} - \partial_{(j}\mathrm{S}'_{i)} \\ &= -8\pi a^2 \Pi^{\mathrm{eff}}_{ij} - 8\pi a^2 \delta\rho_{\mathrm{N}}\mathrm{h}_{ij} + 4(\partial^k\partial_{\langle i}\mathrm{U})\mathrm{h}_{j\rangle k} + 2(\mathcal{D}_{ij}\mathrm{U})(\phi + \psi) + \mathcal{O}(\eta^5)\,, \quad (6.52)\end{aligned}$$

where angle brackets around indices indicate a symmetric and trace-free operation has been used, so that $\mathcal{T}_{\langle ij\rangle} \equiv \mathcal{T}_{(ij)} - \frac{1}{3}\delta_{ij}\mathcal{T}_{kk}$ for any field $\mathcal{T}_{ij}$. This equation can be used to determine the tensor part of the gravitational field, $\mathrm{h}_{ij}$. It also has an effective fluid source, $\Pi^{\mathrm{eff}}_{ij}$, which this time acts as an effective anisotropic stress and is formed from the quadratic contractions of the lower-order small-scale potentials, see Eq. (6.56). Again, the nonlinear structure on small scales couples the large-scale scalar and tensor parts of the cosmological gravitational fields, and again we have additional terms on the right-hand-side that are linear in $\mathrm{h}_{ij}$, resulting in mode-mixing.

As promised, the effective fluid quantities in the perturbation equations above are given as follows:

$$\delta\rho_{\rm eff} = (\bar{\rho} + \bar{p} + \delta\rho_{\rm N})(v_{\rm N})^2 - \frac{1}{\pi a^2}U\nabla^2 U + \frac{3}{4\pi a^2}\Big(\mathcal{H}^2 U + \mathcal{H}U' - \frac{1}{2}(\nabla U)^2\Big) \tag{6.53}$$

$$\delta p_{\rm eff} = \frac{1}{3}(\bar{\rho} + \bar{p} + \delta\rho_{\rm N})(v_{\rm N})^2$$
$$\frac{1}{4\pi a^2}\Big(U'' + 3\mathcal{H}U' - \frac{7}{6}(\nabla U)^2 + a^2 U(\Lambda - 8\pi\bar{p})\Big) + \frac{1}{3\pi a^2}U\nabla^2 U \tag{6.54}$$

$$Q_i^{\rm eff} = \big(\bar{\rho} + \bar{p} + \delta\rho_{\rm N}\big)v_{{\rm N}i} + \frac{1}{4\pi a^2}\partial_i(U' + \mathcal{H}U) \tag{6.55}$$

$$\Pi_{ij}^{\rm eff} = (\bar{\rho} + \bar{p} + \delta\rho_{\rm N})v_{{\rm N}\langle i}v_{{\rm N}j\rangle} - \frac{1}{4\pi a^2}\partial_{\langle i}U\partial_{j\rangle}U - \frac{1}{2\pi a^2}U\mathcal{D}_{ij}U\,. \tag{6.56}$$

It can be seen that each of these quantities was constructed only from variables that correspond to small-scale gravitational fields, or background quantities (which will be shown later to be calculated from the average of small-scale quantities). We therefore have a hierarchy of equations that can be solved order-by-order: first, the Friedmann and Newtonian equations (6.45), (6.46) and (6.47), and then the equations that contain large-scale perturbations (6.48)-(6.52). The former of these sets are already calculated routinely in modern cosmological N-body simulations. The latter are modified versions of the usual cosmological perturbation equations on large scales, and can be used to find post-Newtonian equations on small scales (as recently solved for numerically in [43, 44, 69, 122]). Finally, note that the above effective quantities, in Eqs. (6.53)-(6.56), contain terms that would normally only be included in second or third order in cosmological perturbation theory. In particular, the term $\delta\rho_{\rm N}v_{{\rm N}\langle i}v_{{\rm N}j\rangle}$ in Eq. (6.56) would appear at third order in standard perturbation theory, but here should be expected to source a gravitational "slip" in the leading-order part of the large-scale physics. Our approach can be compared to the effective fluid approach studied previously in [41], as well as the large and small wavelength split used in [36, 37].

As a final comment, before moving on to explain the origin of these equations and give detailed explanations of the gauge invariant quantities involved, we note that the usual trick of separating equations like (6.50) and (6.52) into scalar, vector and tensor parts is much more difficult to apply here. This is due to the fact that terms like $(\mathcal{D}_{ij}U)(\phi + \psi)$ do not have scalar, vector and tensor parts that are easy to identify. This term, for example, is a scalar multiplied by a tensor, and in general should be expected to contain scalar, vector and tensor parts. This does not mean that such a separation is impossible – indeed we very much expect it to be possible. It just means that the resulting equations are very messy to write down, which is the reason why we have chosen to present these equations without such a decomposition. The reader should also be warned that manipulation of these equations is considerably more difficult than in either cosmological perturbation theory or standard post-Newtonian theory. This is due to different derivative operators changing the order to the terms they operate on in different ways. This will be made clearer in the sections that follow.

These equations suggest that it may in fact be possible to generate vector and tensor modes from scalar fluctuations, which is already well known in second-order cosmological perturbation theory [128, 130], but is not usually seen at first order. One should also note that these terms, for example $\frac{8\pi a^2}{3}\delta\rho_N(\psi-\phi)$, also mean that Fourier modes no longer decouple in a trivial way as they do in standard first-order perturbation theory, even if no mode-mixing occurs. This is because the Fourier transforms of such terms are expressible only in terms of a convolution integral over all Fourier modes.

## 6.6. Conservation Equations

In this section we will present the stress-energy conservation equations for our two-parameter perturbation theory. This presentation will differ from the procedure used in standard cosmological perturbation theory, in which the linear-order conservation equations can be derived by straightforward manipulation of the linear-order field equations. Instead, we must take into account the fact that derivatives can change the size of objects they act upon in order to gain the correct equations. This complicates the situation considerably. The equations we derive in this section are only directly applicable to the case of a single self-gravitating fluid, although multiple fluid generalisations are possible.

In general, Einstein's equations contain four constraint equations and six evolution equations [45]. This number can be reduced in situations of high symmetry, such as in Friedmann-Lemaître-Robertson-Walker space-times where there is one constraint equation ($\mathcal{H}^2 = \dots$) and one evolution equation ($\mathcal{H}' = \dots$). In this sense, one can identify Eqs. (C.17) and (6.47) as constraint equations, and Eq. (C.16) as an evolution equation. Likewise, at higher order, one can identify Eqs. (6.49), (6.50) & (6.51) as constraint equations, and Eqs. (6.48) as an evolution equation. Equation (6.52) contains both evolution equations for $h_{ij}$ and $S_j$, but also a constraint on the scalar combination $\psi-\phi$. Normally these equations could be separated out from one another by taking spatial divergences; however, we must proceed with a little more caution than usual here due to the differing way in which the spatial derivative operator acts on post-Newtonian and cosmological terms.

We can say that a set of constraint equations is maintained under evolution if, after differentiating with respect to time and substituting from the evolution equations, the same set of equations is recovered. This is an important property for a physical system to have, as it demonstrates that the system is neither overdetermined (a property that would result in different or additional constraint equations being generated at later times), nor underdetermined. In the analysis that follows we will verify that the constraint equations from our two-parameter expansion are, in fact, consistently maintained under evolution. This will also serve as a check on the perturbed stress-energy conservation equations in this formalism, which can of course also be obtained from expanding $\nabla_\mu T^\mu{}_\nu = 0$.

The derivation of our conservation equations will be presented in terms of the following gauge-invariant matter perturbations:

$$\{\delta\rho_{\mathrm{S}}, \delta\rho_{\mathrm{M}}, \delta\rho, \delta\mathrm{p}, \mathrm{v}_{\mathrm{N}i}, \mathrm{v}_{\mathrm{M}i}, \mathrm{v}_i\}\,, \tag{6.57}$$

where $\delta\rho_{\mathrm{S}} = \delta\boldsymbol{\rho}^{(0,2)}$ is the part of $\delta\rho_{\mathrm{N}}$ that varies over short scales, and $\delta\rho_{\mathrm{M}} = \boldsymbol{\rho}^{(1,1)}$ is the mixed part that is perturbed in both $\epsilon$ and $\eta$. This gives $\delta\rho_{\mathrm{N}} = \delta\rho_{\mathrm{S}} + \delta\rho_{\mathrm{M}}$ as the source term of Eq. (6.47). Here we take the pressure term to be given by $\delta\mathrm{p} = \mathbf{P}^{(1,0)} + \mathbf{P}^{(1,2)} + \frac{1}{2}\mathbf{P}^{(0,4)} + \dots$, where the ellipsis denotes higher-order terms that will be required in the manipulations that follow. Finally, we also introduce the mixed order velocity field $\mathrm{v}_{\mathrm{M}i} = v_i^{(1,1)}$, as the natural two-parameter extension of the perturbed peculiar velocity. Whilst this term does not appear in the gravitational field equations at $\mathcal{O}(\eta^4 L_N^{-2})$, it is found to be required when one considers the stress-energy conservation equations. The other terms are all as defined above.

The corresponding set of gauge-invariant metric perturbations are given by

$$\{\mathrm{U}_{\mathrm{S}}, \mathrm{U}_{\mathrm{M}}, \phi, \psi, \mathrm{B}_i, \mathrm{A}_i, \mathrm{h}_{ij}\}\,, \tag{6.58}$$

where $\mathrm{U}_{\mathrm{S}} = -\frac{1}{2}h_{00}^{(0,2)}$ and $\mathrm{U}_{\mathrm{M}} = -\frac{1}{2}h_{00}^{(1,1)}$ are the short-wavelength and mixed-order parts of $U$, defined such that $\mathrm{U} = \mathrm{U}_{\mathrm{S}} + \mathrm{U}_{\mathrm{M}}$. We have also introduced the vector gravitational potential $\mathrm{A}_i = -(h_{0i}^{(1,0)} + h_{0i}^{(1,2)})$, such that the full vector potential can be written $\mathrm{S}_i = \mathrm{B}_i + \mathrm{A}_i$. Again, the other potentials are defined as in the previous section.

The new quantities introduced in Eqs. (6.57) and (6.58) are motivated by close examination of the conservation equation, as presented below, after which it becomes clear that the evolution equations satisfied by $\delta\rho_{\mathrm{S}}$ and $\delta\rho_{\mathrm{M}}$ take different forms from one another. This motivates us to separate out the corresponding pairs of gravitational potentials $\{\mathrm{U}_{\mathrm{S}}, \mathrm{U}_{\mathrm{M}}\}$ and $\{\mathrm{B}_i, \mathrm{A}_i\}$. Furthermore, the mixed order peculiar velocity, $\mathrm{v}_{\mathrm{M}i}$, does not actually appear in any of the field equations (6.48)-(6.52), but is required to provide a complete set of closed conservation equations. Having introduced these variables for the purpose of performing calculations, where possible we will present the final Euler equations in terms of the original variables used in Eqs. (6.48)-(6.52).

### 6.6.1. Conservation of the Friedmann and Newton-Poisson equations

Let us start with the background Friedmann equations, to elucidate the concept. Taking the time derivative of the two-parameter Friedmann equation gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{H}^2) = 2\mathcal{H}\mathcal{H}' = \frac{8\pi a^2}{3}\left(\bar{\rho}' + 2\mathcal{H}\bar{\rho}\right) + \frac{2}{3}\Lambda\mathcal{H}a^2 \; . \tag{6.59}$$

Now using the two-parameter Raychaudhuri equation to eliminate $\mathcal{H}'$ gives

$$2\mathcal{H}\mathcal{H}' = -\frac{8\pi a^2}{3}(\mathcal{H}\bar{\rho} + 3\mathcal{H}\bar{\mathrm{p}}) + \frac{2}{3}\Lambda\mathcal{H}a^2 \; .$$

As usual, this equation shows that the Friedmann equation is recovered if and only if

$$\boxed{\bar{\rho}' + 3\mathcal{H}(\bar{\rho} + \bar{\mathrm{p}}) = 0} \; , \tag{6.60}$$

which can be straightforwardly verified to be the time component of the stress-energy conservation equation. This is exactly the same as the energy conservation equation from standard Friedmann cosmology, although here the background energy density $\bar{\rho}$ should be understood as the average of the Newtonian mass, which is formally part of the perturbative expansion performed on small scales. This already shows an interesting link between the gravitational fields on large and small scales, which was exploited in Ref. [133] to find consistency relations between super and sub-horizon gravitational potentials.

We can now repeat this procedure for the scalar gravitational potential $\mathrm{U}_{\mathrm{S}}$. Differentiating the leading-order part of Eq. (6.47) with respect to conformal time gives

$$\nabla^2 \mathrm{U}'_{\mathrm{S}} = 4\pi a^2 (\delta\rho'_{\mathrm{S}} + 2\mathcal{H}\delta\rho_{\mathrm{S}}) \, .$$

It can be seen that taking the spatial derivative of equation (6.51) will result in another term $\nabla^2 \mathrm{U_S}'$, which can be used to cancel the appearance of this term in the equation above. Explicitly, we obtain

$$\partial^i \nabla^2 \mathrm{B}_i + 4\nabla^2(\mathrm{U}'_{\mathrm{S}} + \mathcal{H}\mathrm{U}_{\mathrm{S}}) + 16\pi a^2 \partial^i \big((\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}i}\big) = 0 \, ,$$

which on noting that $\partial^i \mathrm{B}_i = 0$ gives

$$\nabla^2(\mathrm{U}'_{\mathrm{S}} + \mathcal{H}\mathrm{U}_{\mathrm{S}}) = -4\pi a^2 \partial^i \big((\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}i}\big) \, .$$

Cancelling $\nabla^2 \mathrm{U_S}'$ from these equations then gives us back the constraint equation (6.47), if and only if

$$\boxed{\delta\rho'_{\mathrm{S}} + 3\mathcal{H}\delta\rho_{\mathrm{S}} = -\partial^i \big((\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}i}\big)} \, , \tag{6.61}$$

which is the Newtonian equation for the conservation of energy on an expanding background. The reader may note that taking the spatial divergence of the vector equation (6.51) does not change the relative size of any terms, as all quantities are either post-Newtonian or mixed in perturbations (i.e. have $n \neq 0$, from Eq. (6.1)). Both equations (6.60) and (6.61) involve quantities of order $\mathcal{O}(\eta^3/L_N^3)$, indicating that the nonlinear leading-order Newtonian fluctuations in the mass density can be of the same size as (or larger than) their mean values. This pleasing feature allows the construction of models where density contrasts on small scales can be very large.

At all subsequent orders, where terms from the post-Newtonian and cosmological sectors of the theory appear concurrently, we must be more careful, as post-Newtonian quantities become smaller by factors of $\eta$ under the action of a time derivative. The result of this is that certain terms are promoted to lower-order by spatial differentiation. This will be very important in obtaining the Euler equations at higher-orders in our expansion: A naive derivation of the same equations, by differentiating the field equations from Section 6.5.2 only, would result in errors.

Let us now consider the time derivative of the vector equation (6.51), which does not contain any terms that change size under the action of a spatial derivative, as it contains post-Newtonian terms only. This gives

$$\nabla^2 \mathrm{B}'_i + 4\partial_i(\mathrm{U}''_\mathrm{S} + \mathcal{H}'\mathrm{U}_\mathrm{S} + \mathcal{H}\mathrm{U}'_\mathrm{S}) + 16\pi a^2\big((\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_\mathrm{S})\mathrm{v}_{\mathrm{N}i}\big)' \\ + 16\pi a^2\big(2\mathcal{H}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_\mathrm{S})\mathrm{v}_{\mathrm{N}i}\big) = 0\,. \tag{6.62}$$

Likewise, the first non-trivial order of the trace-free field equation (6.52) is

$$\mathcal{D}_{ij}(\phi - \psi) - 2\mathcal{H}\partial_{(j}\mathrm{A}_{i)} - \partial_{(j}\mathrm{A}'_{i)} - 2\mathcal{H}\partial_{(j}\mathrm{B}_{i)} - \partial_{(j}\mathrm{B}'_{i)} + \nabla^2\mathrm{h}_{ij} - \mathrm{h}''_{ij} - 2\mathcal{H}\mathrm{h}'_{ij} \\ = -8\pi a^2\Pi^{\mathrm{eff}}_{ij} - 8\pi a^2\delta\rho_\mathrm{S}\mathrm{h}_{ij} + 4(\partial^k\partial_{\langle i}\mathrm{U}_\mathrm{S})\mathrm{h}_{j\rangle k} + 2(\mathcal{D}_{ij}\mathrm{U}_\mathrm{S})(\phi + \psi)\,. \tag{6.63}$$

Taking the leading-order part of the divergence of this equation, and using the result

$$\partial^k\partial^j\big(\partial_{\langle i}\mathrm{U}_\mathrm{S}\;\mathrm{h}_{j\rangle k}\big) = \frac{1}{6}\partial_i\mathcal{D}^{jk}\mathrm{U}_\mathrm{S}\mathrm{h}_{jk} + 2\pi a^2(\partial^k\delta\rho_\mathrm{S})\mathrm{h}_{ik} - \frac{1}{3}\delta_{ij}\partial^j(\mathcal{D}^{lk}\mathrm{U}_\mathrm{S})\mathrm{h}_{lk}\,, \tag{6.64}$$

we obtain

$$\frac{2}{3}\nabla^2\partial_i(\phi - \psi) - \frac{1}{2}\nabla^2(\mathrm{B}'_i + \mathcal{H}\mathrm{B}_i) = -\,8\pi a^2\partial^j\Pi^{\mathrm{eff}}_{ij} + \frac{2}{3}(\partial_i\mathcal{D}^{kj}\mathrm{U}_\mathrm{S})\mathrm{h}_{kj} \\ + \frac{16\pi a^2}{3}(\partial_i\delta\rho_\mathrm{S})(\phi + \psi)\,. \tag{6.65}$$

We can proceed further by looking at the leading-order parts of the spatial gradients of the scalar gravitational field equations (6.48)-(6.49). These can be combined to obtain

$$4\pi a^2(\partial_i\delta\mathrm{p} + \partial_i\delta\mathrm{p}_{\mathrm{eff}}) + \frac{1}{3}(\partial_i\mathcal{D}^{kj}\mathrm{U}_\mathrm{S})\mathrm{h}_{kj} + \frac{8\pi a^2}{3}(\partial_i\delta\rho_\mathrm{S})(\phi + \psi) = \frac{1}{3}\nabla^2\partial_i(\phi - \psi)\,. \tag{6.66}$$

Substituting (6.66) into (6.65), and using Eq. (6.51), then yields the following expression for $\nabla^2\mathrm{B}'_i$:

$$\nabla^2\mathrm{B}'_i = \;16\pi a^2(\partial_i\delta\mathrm{p} + \partial^j\Pi^{\mathrm{eff}}_{ij} + \partial_i\delta\mathrm{p}_{\mathrm{eff}}) + 4\partial_i(2\mathcal{H}\mathrm{U}'_\mathrm{S} + 2\mathcal{H}^2\mathrm{U}_\mathrm{S}) \\ + 16\pi a^2\Big(2\mathcal{H}\big(\mathrm{v}_{\mathrm{N}i}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_\mathrm{S})\big)\Big)\,. \tag{6.67}$$

Now, substituting this into Eq. (6.62) we obtain

$$0 = 4\partial_i\big(\mathrm{U}''_\mathrm{S} + 3\mathcal{H}\mathrm{U}'_\mathrm{S} + (2\mathcal{H}^2 + \mathcal{H}')\mathrm{U}_\mathrm{S}\big) + 16\pi a^2\big(\partial^j\Pi^{\mathrm{eff}}_{ij} + \partial_i\delta\mathrm{p} + \partial_i\delta\mathrm{p}_{\mathrm{eff}}\big) \\ + 16\pi a^2\Big(\big(\mathrm{v}_{\mathrm{N}i}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_\mathrm{S})\big)' + 4\mathcal{H}\big(\mathrm{v}_{\mathrm{N}i}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_\mathrm{S})\big)\Big)\,. \tag{6.68}$$

Next, we can use the following relation derived from taking spatial derivatives of the effective fluid quantities:

$$
\begin{aligned}
16\pi a^2(\partial^j \Pi^{\mathrm{eff}}_{ij} + \partial_i \delta \mathrm{p}_{\mathrm{eff}}) = \; & 16\pi a^2 \partial^j \Big( \mathrm{v}_{\mathrm{N}i}\mathrm{v}_{\mathrm{N}j}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) \Big) \\
& - 4\partial_i \big( \mathrm{U}''_{\mathrm{S}} + 3\mathcal{H}\mathrm{U}'_{\mathrm{S}} + (2\mathcal{H}' + \mathcal{H}^2)\mathrm{U}_{\mathrm{S}} \big) \; . \qquad (6.69)
\end{aligned}
$$

This, in conjunction with the spatially averaged Newtonian field equations (6.45) and (6.46), allows Eq. (6.68) to be written as

$$
\begin{aligned}
& \big( \mathrm{v}_{\mathrm{N}i}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) \big)' + \partial^j \big( \mathrm{v}_{\mathrm{N}i}\mathrm{v}_{\mathrm{N}j}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) \big) + \partial_i \delta \mathrm{p} \\
& + 4\mathcal{H}\big( \mathrm{v}_{\mathrm{N}i}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) \big) + (\partial_j \mathrm{U}_{\mathrm{S}})(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) = 0 \; . \qquad (6.70)
\end{aligned}
$$

Further simplification can be made using the Newtonian-level energy conservation equations (6.60) and (6.61) to obtain the more familiar form.

$$
\begin{aligned}
& \mathrm{v}'_{\mathrm{N}j}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) + \mathrm{v}_{\mathrm{N}i}\partial^i \mathrm{v}_{\mathrm{N}j}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) + \mathrm{v}_{\mathrm{N}j}\bar{\mathrm{p}}' + \mathrm{v}_{\mathrm{N}j}\mathcal{H}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) \\
& \quad = -(\partial_j \mathrm{U}_{\mathrm{S}})(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) - \partial_j(\delta \mathrm{p}) \; . \qquad (6.71)
\end{aligned}
$$

We have checked that this is precisely the generalisation of the Euler equation that is derived by direct calculation of the spatial components of the stress-energy conservation equations up to terms $\mathcal{O}(\eta^4/L_N^3)$, as expected from standard post-Newtonian theory on an expanding background. The only addition here is a time-dependent background pressure that would be considered negligible in typical post-Newtonian systems [50] but could be relevant in cosmological systems [77]. Whilst this is the expected result, it is notable that within this formalism, we had to take cosmological quantities into account to derive it. It is significant that this result from Newtonian gravity continues to hold even in the presence of long-wavelength cosmological perturbations, and may provide justification for the use of Newtonian simulations to analyse short scale without consideration of the large scale universe.

Finally, we can obtain the conservation equation for $\delta\rho_{\mathrm{M}}$ by taking the next-to-leading-order part of the spatial divergence of the vector gravitational field equation (6.50). This gives

$$
\begin{aligned}
& 16\pi a^2(\partial^i \delta\rho_{\mathrm{S}})(\mathrm{v}_i - \mathrm{A}_i) + 16\pi a^2 \partial^i \mathrm{v}_{\mathrm{P}i}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}) + 16\pi a^2 \partial^i \mathrm{Q}^{\mathrm{eff}}_i \\
& \quad = -\partial^i(8\pi a^2 \delta\rho_{\mathrm{S}})\mathrm{A}_i - 2\mathrm{A}^j \partial_j \nabla^2 \mathrm{U}_{\mathrm{S}} \; , \qquad (6.72)
\end{aligned}
$$

where $\mathrm{v}_{\mathrm{P}i} = v_i^{(0,2)}$ is the first post-Newtonian correction to the peculiar velocity.

On using the leading-order part of the Newton-Poisson equation (6.47), this result simplifies down to $\partial^i \mathrm{Q}_i^{\rm eff} = -(\partial^i \delta\rho_{\rm S})\mathrm{v}_i$. Using the definition of $\mathrm{Q}_i^{\rm eff}$ from Eq. (6.55), and the next-to-leading-order part of Eq. (6.47), we then find

$$\boxed{\delta\rho'_{\rm M} + 3\mathcal{H}\delta\rho_{\rm M} = -\partial^i \mathrm{v}_{{\rm P}i}(\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\rm S}) - (\partial^i \delta\rho_{\rm S})\mathrm{v}_i - \partial^i(\delta\rho_{\rm M}\mathrm{v}_{{\rm N}i})} \, . \tag{6.73}$$

This is the first conservation equation we have found that explicitly links post-Newtonian, mixed, and cosmological quantities. It has no analogue in either post-Newtonian gravity or cosmological perturbation theory, as it involves terms from both such expansion schemes, and is therefore the first term to describe the effect of the interactions between these two sectors on the evolution of the energy density. We have again verified that this equation can be directly obtained by expanding the time component of the stress-energy conservation equation up to terms $\mathcal{O}(\eta^4/L_N^3)$. We referred to this equation earlier when we discussed Equation (6.47). We can regard the additional source terms on the right hand side of this equation as being *post-Newtonian corrections*, induced by the presence of a long wavelength cosmological perturbation. This is generically to be expected as this conservation equation is $\sim \mathcal{O}(\eta^4)$, i.e. it has the same order of magnitude as the gravitational field equations describing long-wavelength cosmological perturbations. A further consequence of this is that it is necessary to solve for these corrections at the same time as solving the gravitational field equations.

The evolution equation for $\delta\rho_{\rm N}$ can be obtained by combining Eqs. (6.61) and (6.73), to get

$$\delta\rho'_{\rm N} + 3\mathcal{H}\delta\rho_{\rm N} = -\partial^i\big((\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\rm N})(\mathrm{v}_{{\rm N}i} + \mathrm{v}_i)\big) + \mathcal{O}(\eta^5) \, , \tag{6.74}$$

where we have used $\delta\rho_{\rm N} = \delta\rho_{\rm S} + \delta\rho_{\rm M}$. Taking the leading-order part of this equation recovers Eq. (6.61), while taking the next-to-leading order gives Eq. (6.73).

### 6.6.2. Conservation of the cosmological perturbation equations

Having verified that the constraints are consistently evolved for the background and Newtonian sectors of the field equations, we now wish to perform the corresponding calculation for the cosmological perturbation equations (6.48)-(6.52). This requires considering terms up to $\mathcal{O}(\eta^5/L_N^3)$. The explicit calculations involved in performing this analysis are somewhat lengthy, and are therefore detailed in Appendix D. Here we will present the results only, in the form of the relevant conservation and Euler equations.

The resultant energy conservation equation for $\delta\rho$ is

$$\begin{aligned}
&\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - 3(\psi' + U_S')(\bar\rho + \bar p + \delta\rho_S) = -\partial^i\big(v_{Mi}(\bar\rho + \bar p + \delta\rho_S)\big)\big) \\
&- \partial^i v_{Ci}(\bar\rho + \bar p + \delta\rho_S - \partial^i\big(v_{Ni}(\delta\rho + \delta p)\big) - v_i\partial^i\delta\rho_M - (\partial^i v_{Pi})\delta\rho_M \\
&- \partial^i\big(v_{Ni}(\bar\rho + \bar p + \delta\rho_S)\big)(\phi + U_S) + \partial^i U_S(\bar\rho + \bar p + \delta\rho_S)v_{Ni} \\
&- \big(v_N^2(\bar\rho + \bar p + \delta\rho_S)\big)' - 4\mathcal{H}v_N^2(\bar\rho + \bar p + \delta\rho_S) - \frac{1}{2}\,\partial_i(v_N^2\, v_N^i(\bar\rho + \bar p + \delta\rho_S))\ , \quad (6.75)
\end{aligned}$$

where $v_{Ci} = v_i^{(1,0)}$. It can be seen that this equation is sourced by typical cosmological perturbation theory terms, such as $3\psi'(\bar\rho + \bar p)$, but also byproducts of leading-order terms such as $\mathcal{H}v_N^2(\bar\rho+\bar p+\delta\rho_S)$ and mixed-order terms like $\partial^i\big((\bar\rho+\bar p+\delta\rho_S)v_{Mi}\big)$. Some care is required in interpreting this equation as $\delta\rho$ has a different meaning when it appears under a spatial gradient, as the cosmological contribution $\boldsymbol{\rho}^{(1,0)}$ gains an extra relative order-of-smallness under the action of a gradient compared to the mixed and post-Newtonian contributions $\boldsymbol{\rho}^{(1,2)}$ and $\boldsymbol{\rho}^{(0,4)}$. Thus, when considering the term $\partial^i\big(v_{Ni}(\delta\rho + \delta p)\big)$ the reader should understand the product term $v_{Ni}\partial^i(\delta\rho + \delta p)$ to include only mixed and post-Newtonian contributions, whilst the product term $(\partial^i v_{Ni})(\delta\rho + \delta p)$ should be understood to include all contributions, since the spatial gradient acting on the Newtonian peculiar velocity does not alter its size. The term $\delta\rho'$ should be understood to include all contributions, as the action of a conformal time derivative does not make any term small or larger than any other, regardless of their origin.

Likewise, the Euler equation for the velocity field $v_i$ is found to be

$$\begin{aligned}
&\big((v_i - A_i)(\bar\rho + \bar p + \delta\rho_S) + v_{Ni}\delta\rho_M\big)' + 4\mathcal{H}\big((v_i - A_i)(\bar\rho + \bar p + \delta\rho_S) + v_{Ni}\delta\rho_M\big) \\
&+ (v_i - A_i)\partial^j\big(v_{Nj}(\bar\rho + \bar p + \delta\rho_S)\big) + v_j\partial^j\big(v_{Ni}(\bar\rho + \bar p + \delta\rho_S)\big) \\
&+ (\partial^j v_{Pj})v_{Ni}(\bar\rho + \bar p + \delta\rho_S) + \partial^j(v_{Ni}v_{Nj}\delta\rho_M) + (\partial^j v_{Pi})v_{Nj}(\bar\rho + \bar p + \delta\rho_S) \\
&= -\partial_i(\phi + U_M)(\bar\rho + \bar p + \delta\rho_S) - \delta\rho_M\partial_i U_S - \partial_i(\delta p)\ . \qquad (6.76)
\end{aligned}$$

This is clearly a vector equation, and the evolution of the irrotational part of $v_i$ and vector gravitational potential $A_i$ can be seen to be given by its divergence and divergence-less parts, respectively. As before, we choose not to do this decomposition explicitly here, as the product terms and the rules associated with derivatives acting on different types of fields will complicate the results. This equation is reminiscent of the corresponding Euler equation from standard cosmological perturbation theory, with extra terms due to the existence of the nonlinear structures on small scales. We note in particular that the mixed term $\partial^j(\mathrm{v}_{\mathrm{N}i}\mathrm{v}_{\mathrm{N}j}\delta\rho_{\mathrm{M}})$ acts as a source for cosmological peculiar velocities and vector gravitational perturbations. We have verified that both Eq. (6.75) and Eq. (6.76) are recovered from the stress-energy conservation equations at $\mathcal{O}(\eta^5/L_N^3)$.

We can combine Eqs. (6.71) and (6.76) to write a multi-order Euler equation for the evolution of the Newtonian and cosmological peculiar velocities;

$$
\begin{aligned}
&\big((\mathrm{v}_{\mathrm{N}i}+\mathrm{v}_i-\mathrm{S}_i)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{N}})\big)' \\
&+4\mathcal{H}\big((\mathrm{v}_{\mathrm{N}i}+\mathrm{v}_i-\mathrm{S}_i)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{N}})\big)+\partial_j(\mathrm{U}+\phi)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{N}}) \\
&+\partial^j\big((\mathrm{v}_{\mathrm{N}i}+\mathrm{v}_i-\mathrm{S}_i)(\mathrm{v}_{\mathrm{N}j}+\mathrm{v}_j)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{N}})\big)+\partial_i\delta\mathrm{p}=0+\mathcal{O}(\eta^6)\,. \qquad (6.77)
\end{aligned}
$$

The evolution equation for $\mathrm{v}_{\mathrm{N}i}$ given in Eq. (6.71) is then recovered by taking the order $\mathcal{O}(\eta^4)$ part of this equation, while Eq. (6.76) is recovered by taking the order $\big(\mathcal{O}(\eta^5)\big)$ part.

The final equation required to complete a closed set of evolution equations for the set $\{\delta\rho_{\mathrm{S}},\delta\rho_{\mathrm{M}},\delta\rho,\delta p,\mathrm{v}_{\mathrm{N}i},\mathrm{v}_{\mathrm{M}i},\mathrm{v}_i\}$ is an evolution equation for $\mathrm{v}_{\mathrm{M}i}$. This is obtained from the perturbed stress-energy conservation equations at $\mathcal{O}(\eta^6/L_N^3)$, and is as follows:

$$
\boxed{
\begin{aligned}
&\big(\mathrm{v}_{\mathrm{N}i}(\delta\rho+\delta\mathrm{p})+(\mathrm{v}_{\mathrm{M}i}-\mathrm{B}_i)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})+(\mathrm{v}_i-\mathrm{A}_i)\delta\rho_{\mathrm{M}}+2\mathrm{h}_{ij}\mathrm{v}_{\mathrm{N}}^j(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big)' \\
&+4\mathcal{H}\big(\mathrm{v}_{\mathrm{N}i}(\delta\rho+\delta\mathrm{p})+(\mathrm{v}_{\mathrm{M}i}-\mathrm{B}_i)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})+(\mathrm{v}_i-\mathrm{A}_i)\delta\rho_{\mathrm{M}} \\
&-\Big(\big(\mathrm{v}_{\mathrm{N}i}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big)'+4\mathcal{H}\mathrm{v}_{\mathrm{N}i}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\Big)(\phi+2\psi+3\mathrm{U}_{\mathrm{S}}) \\
&+2\mathrm{h}_{ij}\mathrm{v}_{\mathrm{N}}^j(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big)-5\mathrm{v}_{\mathrm{N}i}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})(\psi'+\mathrm{U}_{\mathrm{S}}') \\
&+\frac{1}{2}\big(\mathrm{v}_{\mathrm{N}}^2\mathrm{v}_{\mathrm{N}i}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big)'+2\mathcal{H}(\mathrm{v}_{\mathrm{N}}^2\mathrm{v}_{\mathrm{N}i}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}}))= \\
&-\partial^j\big(\mathrm{v}_{\mathrm{N}j}(\mathrm{v}_{\mathrm{M}i}-\mathrm{B}_i)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big)-\partial^j\big(\mathrm{v}_{\mathrm{N}i}\mathrm{v}_{\mathrm{M}j}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big)-(\mathrm{v}_i-\mathrm{A}_i)\mathrm{v}_j\partial^j\delta\rho_{\mathrm{S}} \\
&-(\mathrm{v}_i-\mathrm{A}_i)\partial^j\mathrm{v}_{\mathrm{P}j}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})-\mathrm{v}_j\partial^j\mathrm{v}_{\mathrm{P}i}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})-\partial^j\big(\mathrm{v}_{\mathrm{N}i}\mathrm{v}_{\mathrm{N}j}(\delta\rho+\delta\mathrm{p})\big) \\
&-\mathrm{v}_{\mathrm{N}j}\partial^j(\mathrm{v}_{\mathrm{C}i}-\mathrm{A}_i)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})-\mathrm{v}_{\mathrm{N}i}\partial^j\mathrm{v}_{\mathrm{C}j}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})-(\mathrm{v}_i-\mathrm{A}_i)\partial^j(\delta\rho_{\mathrm{M}}\mathrm{v}_{\mathrm{N}j}) \\
&-\mathrm{v}_j\partial^j(\delta\rho_{\mathrm{M}}\mathrm{v}_{\mathrm{N}i})-(\partial^j\mathrm{v}_{\mathrm{P}j})\delta\rho_{\mathrm{M}}\mathrm{v}_{\mathrm{N}i}-2\mathrm{h}_{ij}\mathrm{v}_{\mathrm{N}}^j\partial^k\big(\mathrm{v}_{\mathrm{N}k}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big) \\
&+4\mathrm{v}_{\mathrm{N}j}\mathrm{v}_{\mathrm{N}i}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\partial^j\mathrm{U}_{\mathrm{S}}+2(\mathrm{U}_{\mathrm{S}}+\psi)\partial^j\big(\mathrm{v}_{\mathrm{N}j}\mathrm{v}_{\mathrm{N}i}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big) \\
&-\mathrm{v}_{\mathrm{N}}^j(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\partial_i(\mathrm{B}_j+\mathrm{A}_j)-(\partial_i\mathrm{U}_{\mathrm{S}})\big(\delta\rho+\delta\mathrm{p}-2(\mathrm{U}_{\mathrm{S}}+\phi)(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\big) \\
&-\delta\rho_{\mathrm{M}}\partial_i(\phi+\mathrm{U}_{\mathrm{M}})-(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\partial_i\phi_{\mathrm{P}}-\partial_i\delta\mathrm{p} \\
&-(\partial^j\mathrm{v}_{\mathrm{P}i})\delta\rho_{\mathrm{M}}\mathrm{v}_{\mathrm{N}j}-2\mathrm{h}_{ik}\mathrm{v}_{\mathrm{N}}^j\partial^k\mathrm{v}_{\mathrm{N}j}(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})-2\mathrm{v}_{\mathrm{N}}^2(\bar\rho+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\partial_i\mathrm{U}_{\mathrm{S}}
\end{aligned}
}
\qquad (6.78)
$$

where $\phi_{\rm P} = \phi^{(0,4)} + \phi^{(1,2)}$. This equation displays further interesting characteristics, for example, coupling between cosmological tensor and Newtonian vector and scalar perturbations. This can be seen in the term $\left(h_{ij} v_{\rm N}^{j}(\bar{\rho} + \bar{p} + \delta\rho_{\rm S})\right)'$, and should be expected to result in new physical effects due to the interplay between perturbations on different length scales. This evolution equation was determined directly from the spatial components of the stress-energy conservation equation, and is required to consistently evolve the source terms in Eq. (6.75), even though it does not appear in the field equations itself (at the order we are considering).

## 6.7. Discussion

In this chapter we have constructed a two-parameter perturbation expansion around an FLRW background that simultaneously describes non-linear structures on small-scales and linear structures on large scales, including matter, radiation and a cosmological constant. In doing so we used both cosmological perturbation theory and the post-Newtonian expansion. As this expansion is able to model large density contrasts and different matter components, it therefore both contains the essential features of the real Universe and has a number of potential advantages over standard cosmological perturbation theory.

We find that the small-scale Newton-Poisson equation for the scalar gravitational potential occurs at the same order in perturbations as the Friedmann equation, but that they can be separated after the introduction of a suitable homogeneity scale. At leading order, this results in a small-scale Newton-Poisson equation sourced by the inhomogeneous part of the Newtonian energy density, and large-scale Friedmann equations sourced by the spatial average of the leading-order parts of the energy density, pressure, and the cosmological constant. Our results give no indications that the effects of small-scale non-linearities should be expected to cause acceleration of the large-scale Universe, but we do find that they should be expected to affect large-scale perturbations. This is because the higher-order field equations include quadratic Newtonian potentials within the effective fluid terms. They therefore contain valuable information about non-linear gravity, and could potentially be used to identify relativistic effects in observations of large-scale structure.

By presenting the higher-order field equations in terms of an effective fluid we are able to highlight the similarities and differences between our formalism and regular cosmological perturbation theory. Our effective fluid description also enables an easier physical interpretation of the effects of non-linearities in the field equations, which clearly lead to (for example) a large-scale effective pressure and anisotropic stress. Since the effective fluid terms are all constructed from the solution to the short-scale Newtonian gravitational potential, their properties should be able to be determined from N-body simulations. Once the form of these effective fluids has been identified, one can proceed to solve the cosmological equations for the long-wavelength perturbations. This method of solution is available to us because of the hierarchical nature of the perturbation equations – short-scale fluctuations appear at lower-order compared to cosmological perturbations, and so can be solved for before cosmological perturbations. Within this prescription we observe a mixing of scales, as well as mode-mixing at what would normally be considered to be linear order in cosmological potentials.

We have derived and presented the relativistic Euler equations that exist in the two-parameters expansion proposed in Refs. [125] and [126]. These equations describe the evolution of density perturbations and peculiar velocities for a self-gravitating perfect fluid in an FLRW background. These equations are written down in gauge-invariant variables, and were used to confirm that the constraint equations from the two-parameter perturbation expansion are consistently evolved, despite the fact that terms can change size under differentiation. This gives confidence that the scheme is internally self-consistent and complete, and can be used to model the relativistic effects of nonlinear structures in perturbation theory.

The resulting Euler equations for the inhomogeneous part of the leading-order matter density and the peculiar velocity, together with the leading order gravitational Poisson equation, reproduce the standard results of Newtonian theory on an expanding background. These leading-order equations have well-known solutions in terms of Green's functions and numerical $N$-body simulations. Subsequent higher-order equations that govern the leading-order contributions to the large-scale gravitational potentials are then given as linear partial differential equations that contain the known solutions to the lower-order Newtonian equations as source terms. In a sense, one can consider the equations for the cosmological quantities as being the result of performing a linear cosmological perturbation theory expansion on a background that is allowed to contain Newtonian gravitational fields (or vice versa). This explicitly shows the link between gravitational fields on large and short scales which occurs due to the nonlinearity of Einstein's equations.

In the subsequent chapters, we will explore methods to approximate solutions to a simplified version of the full two-parameter perturbation theory scenario. Approximations to solutions to the leading order Newtonian Euler equations can be obtained by applying Eulerian perturbation theory [84–86], allowing corresponding approximations to be applied to the large scale dynamics.

# 7. Approximate solutions to 2PPT in dust-dominated universes

In this chapter, we will consider the use of Newtonian perturbation theory to construct approximate solutions to the scalar two-parameter perturbation theory (2PPT) equations in a restricted scenario where vector and tensor degrees of freedom are neglected, and the leading order homogeneous geometry is Einstein-de Sitter. The work in this chapter is of my own doing, and is based heavily on the material presented in the paper [134]. The approximate solutions are used to calculate the leading order dark-matter bispectrum, which is directly compared to the analogous result in second-order cosmological perturbation theory.

## 7.1. Motivation

In the previous chapter, we presented a formalism capable of accurately modelling a universe exhibiting short-scale nonlinear structure simultaneously with long-wavelength linear perturbations. The short-scale nonlinear structure was shown to be accurately described by the equations of Newtonian cosmology, whilst the long-wavelength linear fluctuations satisfied a modified set of first-order perturbation equations.

Unfortunately, the features that rendered the cosmological equations in the two-parameter theory interesting also present severe challenges when it comes to finding solutions. In particular, one is forced to consider spatially inhomogeneous linear differential operators at leading-order in cosmological perturbations (something that does not occur at all in regular cosmological perturbation theory). This makes eigenfunctions difficult to find, as they are dependent on the nonlinear solutions to the Newtonian equations, which are themselves dependent on spatially inhomogeneous and stochastic initial data. Added to this, we have the extra complication that taking derivatives in this formalism is non-trivial, as space and time derivatives do not act in the same way in the two different sectors of the theory.

In this chapter, we consider the problem of finding solutions to the equations of two-parameter perturbation theory in an Einstein-de Sitter universe, and using them to calculate the bispectrum of matter. This is achieved by the key assumption that we can use Eulerian perturbation theory in the quasi-linear regime, in order to find approximate solutions to the leading-order post-Newtonian equations. Each two-parameter perturbation in the system is then expanded using the same approach, leading to a hierarchy of linear equations that can be solved order-by-order to get successively more accurate approximations to the original two-parameter equations. The solutions obtained can then be used to calculate the statistical properties of the matter distribution, and hence observables.

In the two-parameter expansion, terms that might traditionally be considered as higher-order are promoted to the same equations as first-order cosmological perturbations due to the effects of short-scale nonlinear structure. There are also interaction terms between unsolved for cosmological perturbations and the short-scale Newtonian fluctuations (e.g. $\rho^{(0,2)}\psi$) - this modifies the structure of the linear partial differential equations. These interaction terms are outside the scope of regular perturbation theory and require careful thought in dealing with. In order to address the issue of the interaction terms, we we will tackle a simplified version of this problem that will illuminate the way forward. We will neglect

- Vector and tensor modes,
- Post-Newtonian and mixed order corrections.

Neither of these assumptions is strictly justified in the two-parameter expansion. Post-Newtonian terms appear in the same field equations as the first order cosmological perturbation equations, and the post-Newtonian correction to the short scale density contrast is formally the same size as the first-order cosmological density contrast. Furthermore, the consistent propagation of the constraint equations as discussed in Chapter 6 requires the generation of vector and tensor modes from scalars, even if initial conditions are set such that vectors and tensors vanish.

In order therefore to justify these assumptions to some extent, we can use the fact that we intend to solve the leading order inhomogeneous Newtonian problem using Newtonian perturbation theory. This approximation limits the scope of our intended application of the two-parameter scheme to situations where the short-scale nonlinear density contrasts are at least *quasi-linear* (i.e $\sigma_{\delta_N} \leq 1$, where $\sigma_{\delta_N}$ is the *variance* of the short-scale Newtonian density contrast). In these scenarios, where the size of the short-scale Newtonian density contrast is to be limited, one would expect the corresponding post-Newtonian corrections to be smaller than they could potentially be in the scenario with fully nonlinear short-scale density contrasts.

With regard to the problem of generation of vector and tensor modes, the use of Newtonian perturbation theory combined with the linearity of the two-parameter equations enables one to address this problem in a fashion that is analogous to second-order cosmological perturbation theory, for example in [135, 136]. In particular, once a solution is obtained for the scalar modes, it is possible to calculate the induced vectors and tensors by inserting this solution back into the field equations and solving order by order in the usual way. Since the coupling terms that generate vectors and tensors are formed from products with the usual Newtonian perturbation theory terms and cosmological terms, and solutions exist for both types of terms (as we demonstrate below), it therefore holds that the same techniques could be applied to calculate induced vectors and tensors. Whilst this problem is not solved in this thesis, it can in principle be carried out using the same techniques presented here.

## 7.2. Reduced field equations

In this section we will present the reduced two-parameter formalism, only considering scalar first-order cosmological perturbations and leading-order scalar Newtonian fluctuations. This can be seen to be equivalent to consideration of the following metric,

$$ds^2 = a^2(\tau)\Big[ -(1-2U-2\phi)d\tau^2 + (1-2U-2\psi)\delta_{ij}dx^i dx^j \Big] \,, \tag{7.1}$$

where $U = -\frac{1}{2}\Phi^{(0,2)} \sim \eta^2$ is the leading order Newtonian gravitational field, and follows the counting scheme for post-Newtonian perturbations outlined in Chapter 6. The cosmological perturbations, $\phi = -\frac{1}{2}\Phi^{(1,0)} \sim \epsilon$ and $\psi - \frac{1}{2}\Psi^{(1,0)} \sim \epsilon$, follow the cosmological perturbation theory counting scheme outlined in Chapter 6. The corresponding stress-energy tensor is given by

$$T^{\mu\nu} = (\rho_{\rm N} + \delta\rho)u^\mu u^\nu \,, \tag{7.2}$$

where $\rho_{\rm N} = \boldsymbol{\rho}^{(0,2)}$, $\delta\rho = \boldsymbol{\rho}^{(1,0)}$, $v_{{\rm N}i} = \mathbf{v}_i^{(0,1)}$ and $v_i = \mathbf{v}_i^{(1,0)}$.

The leading-order field equations are given at order $\sim \eta^2/L_{\rm N}^2$ by

$$\mathcal{H}' = -\frac{4\pi a^2}{3}\rho_{\rm N} - \frac{1}{3}\nabla^2 U \ , \tag{7.3}$$

$$\mathcal{H}^2 = \frac{8\pi a^2}{3}\rho_{\rm N} + \frac{2}{3}\nabla^2 U \ , \tag{7.4}$$

where $\mathcal{H} = a'/a$ is the conformal Hubble rate, and primes denote differentiation with respect to conformal time, $\tau$. By averaging these equations it can be seen that we obtain

$$\mathcal{H}' = -\frac{4\pi a^2}{3}\bar{\rho} \ , \tag{7.5}$$

$$\mathcal{H}^2 = \frac{8\pi a^2}{3}\bar{\rho} \ , \tag{7.6}$$

which leaves the fluctuations around the average given by

$$\nabla^2 U = 4\pi a^2 \delta\rho_{\rm N} \ , \tag{7.7}$$

where $\bar{\rho}$ denotes the mean of $\rho_{\rm N}$, and $\delta\rho_{\rm N}$ denotes the fluctuation around the mean. This average value of $\rho_{\rm N}$ must be the same at all points in the Universe, otherwise these equations are inconsistent with the initial assumption of a background FLRW metric with $a = a(\tau)$.

Equations (7.5) and (7.6) are identical to the Friedmann equations for an Einstein-de Sitter (EdS) universe, and admit the well-known solution

$$a = \tau^2 \ , \tag{7.8}$$

$$\text{which implies} \quad \mathcal{H} = \frac{2}{\tau} \ . \tag{7.9}$$

Likewise, Eq. (7.7) can be seen to be identical to the Poisson equation of Newtonian gravity on an expanding background, and correspondingly the solutions for $U$ must be given by the linear sum of Newtonian gravitational potentials of all matter fields.

As was demonstrated in Chapter 6, conservation of the Einstein constraint equations under time evolution demands that $\delta_{\rm N} \equiv \delta\rho_{\rm N}/\bar{\rho}$ and $v_{{\rm N}i}$ satisfy the continuity equation and Euler equations:

$$\delta_{\rm N}' + \partial^i\big(v_{{\rm N}i}\big(1+\delta_{\rm N}\big)\big) = 0 \ , \tag{7.10}$$

$$v_{{\rm N}i}' + \mathcal{H}v_{{\rm N}i} + \partial_i U + v_{{\rm N}j}\partial^j v_{{\rm N}i} = 0 \ . \tag{7.11}$$

Under the additional assumption of vanishing vorticity, these expressions form a closed *nonlinear* system for the three Newtonian perturbations $\{U, \delta_{\rm N}, v_{{\rm N}i}\}$. Their solutions should be understood as the leading-order contribution to the PN sector of the theory, with subsequent higher-order corrections representing relativistic effects. Solutions to this system (for a given initial matter distribution) are usually obtained using either Newtonian N-body simulations, or Newtonian perturbation theory (or "NPT" for short). We will use the latter in this study, though the reader may wish to keep in mind that an all-orders resummed NPT solution still only constitutes the leading-order contribution to the gravitational field in the 2PPT set up.

The next order of field equations is at $\sim \eta^4/L_{\rm N}^2$. Neglecting vectors and tensors, the evolution equation for the scalar degree of freedom and the trace-free $ij$ field equation give

$$(\psi+U)''+3\mathcal{H}(\psi+U)' = \frac{4\pi a^2\bar{\rho}}{3}(1+\delta_{\rm N})v_{\rm N}^2+\mathcal{H}(\psi'-\phi')+\frac{1}{3}\nabla^2(\psi-\phi) \\ +\frac{7}{6}(\nabla U)^2+\frac{2}{3}(\phi+\psi+2U)\nabla^2 U\ , \tag{7.12}$$

and

$$\partial^i\partial_j(\psi-\phi)+2\partial^i U\partial_j U+2(\psi+\phi+2U)\partial^i\partial_j U \\ -\frac{1}{3}\delta^i{}_j\left[\nabla^2(\psi-\phi)+2(\nabla U)^2+2(\psi+\phi+2U)\nabla^2 U\right] \\ = \quad 8\pi a^2\bar{\rho}\,(1+\delta_{\rm N})\big(v_{\rm N}^i v_{{\rm N}j}-\frac{1}{3}\delta^i{}_j v_{\rm N}^2\big)\ , \tag{7.13}$$

while the generalised Poisson and momentum constraint equations give

$$\frac{1}{3}\nabla^2\psi-\mathcal{H}(\psi'+U')-\mathcal{H}^2(\phi+U) = \frac{4\pi a^2\bar{\rho}}{3}\delta+\frac{4\pi a^2\bar{\rho}}{3}(1+\delta_{\rm N})v_{\rm N}^2 \\ -\frac{1}{2}(\nabla U)^2-\frac{4}{3}(\psi+U)\nabla^2 U\ , \tag{7.14}$$

$$\partial_i\big(\psi'+\mathcal{H}\phi\big) = -\frac{3\mathcal{H}^2}{2}(1+\delta_{\rm N})v_i\ \ , \tag{7.15}$$

where $\delta=\frac{\delta\rho}{\bar{\rho}}$ is the cosmological matter density contrast fluctuation divided by the homogeneous component of the leading order Newtonian matter. We have explicitly written out the terms that were previously included within the effective fluid variables $\rho_{\rm eff}$, $\delta p_{\rm eff}$, $\Pi_{\rm eff}$ and $Q_{{\rm eff}\,i}$. These equations can be seen to contain quadratic and even cubic products of lower-order perturbations, as well as products of (unsolved-for) cosmological perturbations and (solved-for) Newtonian perturbations, in ways that simply cannot occur in standard CPT.

For the rest of this thesis, we will refer to Eqs. (7.12)–(7.15) as the *2PPT field equations*. Although these equations are not the only field equations that can be derived using the 2PPT formalism, they do contain the critical physics that the formalism seeks to investigate; the effects of small-scale nonlinearities on large-scale cosmological perturbations. In fact, one could think of these equations as a set that describe first-order cosmological perturbations on top of a universe that already contains nonlinear structure on small scales. In this sense, they model cosmological back-reaction of small-scale structure on the large-scale Universe, within a well-defined framework (although they do not model back-reaction of perturbations onto the actual background expansion).

## 7.3. The utility of Newtonian perturbation theory

Equations (7.12) and (7.13) are difficult to solve. There are a number of reasons for this, including the fact that $\delta_{\mathrm{N}}$, $U$ and $v_{\mathrm{N}i}$ are themselves the solutions to non-linear differential equations (the Eulerian equations of fluid dynamics), and as such are complicated functions of stochastic initial conditions. This renders the linear differential operators on the left-hand sides of these equations dependent on spatial position in a stochastic fashion, which in turn makes it is unclear what set of eigenfunctions should be used as a basis for constructing solutions.

We may compare this to the situation in CPT, where the first-order equations can be expressed heuristically as

$$\hat{\mathcal{L}}_{\mathrm{CPT}}(\tau)\,\boldsymbol{u}_1 = 0\,, \tag{7.16}$$

where $\hat{\mathcal{L}}_{\mathrm{CPT}}(\tau)$ is a matrix-valued linear differential operator containing both spatial and temporal derivatives, but which functionally depends only on conformal time. The $\boldsymbol{u}_1$ in this equation is intended to denote all first-order quantities ($\phi_1$, $\psi_1$, $\delta_1$, $\ldots$) from the relativistic perturbation theory developed in Chapter 4 arranged into a column vector.

$$\boldsymbol{u}_1 = \begin{pmatrix} \phi_1(x,\tau) \\ \psi_1(x,\tau) \\ \delta_1(x,\tau) \\ \vdots \end{pmatrix}\,. \tag{7.17}$$

This homogeneous matrix-valued differential equation can easily be diagonalised in either real space or Fourier space, as $\hat{\mathcal{L}}_{\rm CPT}(\tau)$ does not depend on space.

Similarly, higher-order CPT equations can be written as

$$\hat{\mathcal{L}}_{\rm CPT}(\tau)\,\boldsymbol{u}_2 \sim \boldsymbol{u}_1^2 \tag{7.18}$$

$$\hat{\mathcal{L}}_{\rm CPT}(\tau)\,\boldsymbol{u}_3 \sim \boldsymbol{u}_1\boldsymbol{u}_2 + \boldsymbol{u}_1^3\,, \tag{7.19}$$

where numerical subscripts denote the order of a quantity in the CPT expansion. The key point to note here is that at each order the linear differential operator $\hat{\mathcal{L}}_{\rm CPT}(\tau)$ remains the same, so successive approximations can be found by identifying particular solutions for given source terms and then simply adding them to the original first-order solution.

It is immediately apparent that two-parameter perturbation theory does not follow this structure: The leading-order evolution equations are *nonlinear*, and the sub-leading field equations (7.12)–(7.15)) cannot be written in the form of Eq. (7.18). Instead, what we have is an equation of the form

$$\begin{aligned}&\hat{\mathcal{L}}_{\rm 2PPT}(\tau, U, \updelta_{\rm N}, \upsilon_{{\rm N}i})\,\boldsymbol{u}_{\eta^4}\\ &\qquad\qquad = \hat{\mathcal{L}}(\tau)\,\boldsymbol{u}_{\eta^4} + \hat{D}(\tau, U, \updelta_{\rm N}, \theta_{\rm N})\,\boldsymbol{u}_{\eta^4} = \boldsymbol{u}_{\eta^2}^2\,,\end{aligned} \tag{7.20}$$

where $\hat{D}(\tau, U, \updelta_{\rm N}, \theta_{\rm N})$ is some matrix operator function describing the coupling of the nonlinear leading order solutions to the subleading order solutions, and where $\boldsymbol{u}_{\eta^2}$ is a column vector of the leading-order nonlinear solutions ($\sim \eta^2/L_{\rm N}^2$) and $\boldsymbol{u}_{\eta^4}$ is a column vector of the sub-leading-order solutions ($\sim \eta^4/L_{\rm N}^2$). It can be seen that the linear operator in this equation, $\hat{\mathcal{L}}_{\rm 2PPT}(\tau, U, \updelta_{\rm N}, \upsilon_{{\rm N}i})$, is a function of the nonlinear solutions to the leading-order field equations, which themselves are complicated functions of stochastic initial conditions.

Solving Eq. (7.20) requires care; the usual strategy for diagonalising the linear operator in equations of this type involves taking spatial derivatives of the trace-free $ij$-field equation (7.13), and using the result to eliminate derivatives of the combination $\psi - \phi$ from the evolution equation (7.12). In the case of 2PPT, however, taking spatial derivatives will affect post-Newtonian and cosmological terms in different ways, as explained in Chapter 6. We must be careful to ensure that this operation is performed consistently, and that product terms that can exist at higher orders do not influence the results.

Let us demonstrate this with an example; differentiating the first term in the trace-free $ij$-field equation (7.13) results in

$$\partial_i \partial^j (\partial^i \partial_j (\psi - \phi)) \sim \nabla^4 (\psi - \phi) \sim \frac{\eta^6}{L_{\mathrm{N}}^4} \,. \tag{7.21}$$

The original equation was order $\sim \eta^4 / L_{\mathrm{N}}^2$, while this term is now at order $\sim \eta^6 / L_{\mathrm{N}}^4$; we say that the spatial derivatives have "promoted" this term to higher order. This is potentially problematic, as terms in the trace-free $ij$-field equation at order $\sim \eta^6 / L_{\mathrm{N}}^2$ will also appear at order $\sim \eta^6 / L_{\mathrm{N}}^4$ after differentiation (e.g. $U \delta \rho_{\mathrm{N}} v_{\mathrm{N}i} v_{\mathrm{N}j}$). Such terms therefore need to be considered at the same time, in any consistent treatment. Similar issues arise when using "inverse Laplacians", as the action of inverse derivatives can also affect a quantity's size.

It is foreseeable that there exist terms in the *undifferentiated* $\mathcal{O}(\frac{\eta^6}{L_{\mathrm{N}}^2})$ trace-free $ij$ field equation, for example

$$U \delta \rho_{\mathrm{N}} v_{\mathrm{N}i} v_{\mathrm{N}j} \sim \frac{\eta^6}{L_{\mathrm{N}}^2} \longrightarrow \partial^i \partial^j \big( U \delta \rho_{\mathrm{N}} v_{\mathrm{N}i} v_{\mathrm{N}j} \big) \sim \frac{\eta^6}{L_{\mathrm{N}}^4} \,, \tag{7.22}$$

which would **not** be promoted under spatial derivatives, such that after taking divergences these terms would appear in the same $\mathcal{O}(\frac{\eta^6}{L_{\mathrm{N}}^4})$ *differentiated* field equation, together with $\frac{2}{3} \nabla^4 \phi$. To make matters worse, there could also exist terms of the form

$$\phi \; \delta\rho_{\mathrm{N}} v_{\mathrm{N}i} v_{\mathrm{N}j} \,, \tag{7.23}$$

which, under spatial differentiation, will have at least one component that remains $\mathcal{O}\left(\frac{\eta^6}{L_{\mathrm{N}}^4}\right)$, i.e

$$\begin{aligned} \partial^i \partial^j \big( \phi \; \delta\rho_{\mathrm{N}} v_{\mathrm{N}i} v_{\mathrm{N}j} \big) = {} & \phi \; \delta\rho_{\mathrm{N}} \theta_{\mathrm{N}} \theta_{\mathrm{N}} + \phi \; (\partial^i \delta\rho_{\mathrm{N}}) v_{\mathrm{N}i} \theta_{\mathrm{N}} \\ & + \phi \; (\partial^j \delta\rho_{\mathrm{N}}) \theta_{\mathrm{N}} v_{\mathrm{N}j} + \phi \; (\partial^i \partial^j \delta\rho_{\mathrm{N}}) v_{\mathrm{N}i} v_{\mathrm{N}j} + \mathcal{O}\left( \frac{\eta^7}{L_{\mathrm{N}}^4} \right) . \end{aligned} \tag{7.24}$$

The aforementioned issue has a knock-on effect when considering trying to reverse the process of differentiation (as is often done via "inverse Laplacians", or in Fourier space in the literature). The application of an "inverse Laplacian" to product terms whose constituent pieces follow different counting schemes is potentially ambiguous; unless a term can be demonstrated to be a direct Laplacian of some other quantity (in which case the size of the resulting object would simply be the quantity of which the original Laplacian was taken), it is possible that the "inverse Laplacian" may in fact involve objects of different sizes. In recognition of this potentially troublesome issue, we will choose to consider *all* terms that could potentially contribute to the fully "inverse Laplacian" operated expression. This corresponds to retaining all the objects at every order generated from taking spatial derivatives.

The net effect of this is that applying $\partial_i\partial^j$ to Eq. (7.13) results in an equation with the schematic form

$$\nabla^4(\phi-\psi)+\mathcal{I}(\tau,x)+\mathcal{T}^{ij}(\tau,x)v_iv_j=\mathcal{S}(\tau,x)\ , \tag{7.25}$$

where $\mathcal{I}(\tau,U,\delta_{\rm N},v_{{\rm N}i},\phi,\psi,v_i,\delta)$ and $\mathcal{T}^{ij}(\tau,U,\delta_{\rm N},v_{{\rm N}i})$ are functions of both Newtonian and cosmological perturbations. It may be noted that no quantities of the form $\mathcal{Q}^i\nabla^2\partial_i\phi\sim\eta^6/L_{\rm N}^4$ appear in this equation, due to the fact that such a term would require $\mathcal{Q}^i\sim\eta/L_{\rm N}$. However, no such term can exist since only $v_{{\rm N}i}$ has the required index structure and magnitude, but $v_{{\rm N}i}$ always appears quadratically in the trace-free part of $T_{ij}$.

We retain all the terms that could possibly contribute to $\mathcal{T}^{ij}$ and $\mathcal{I}$, which include orders $\sim\eta^4/L_{\rm N}^4$, $\sim\eta^5/L_{\rm N}^4$ and $\sim\eta^6/L_{\rm N}^4$.

Equation (7.25) is nonlinear in $v_i$, and has a particularly complex operator structure (differential operators depend inhomogeneously on the leading-order solutions of the nonlinear Eulerian equations). This means that simply applying an inverse Laplacian, as one might do in CPT, will not be sufficient here.

In order to proceed analytically, it is therefore useful to make further approximations. To this end, we will use NPT to solve the continuity and Euler equations (7.10)-(7.11). This works by performing a series expansion on all quantities in the Newtonian density contrast $\delta_{\rm N}$, which in the present case can be equivalently given in terms of the seed gravitational potential, $\varphi$. This means that we write, for example, the Newtonian gravitational potential as

$$\delta_{\rm N}=\delta_{\rm N}^{(1)}+\frac{1}{2}\delta_{\rm N}^{(2)}+\cdots=\sum_{n=1}^{\infty}\frac{\delta_{\rm N}^{(n)}}{n!} \tag{7.26}$$

$$\theta_{\rm N}=\theta_{\rm N}^{(1)}+\frac{1}{2}\theta_{\rm N}^{(2)}+\cdots=\sum_{n=1}^{\infty}\frac{\theta_{\rm N}^{(n)}}{n!} \tag{7.27}$$

$$U=U^{(1)}+\frac{1}{2}U^{(2)}+\cdots=\sum_{n=1}^{\infty}\frac{U^{(n)}}{n!}\ , \tag{7.28}$$

where these quantities are understood to be the the linear and second-order parts respectively from NPT, and where the superscript here denotes the order in $\varphi$.

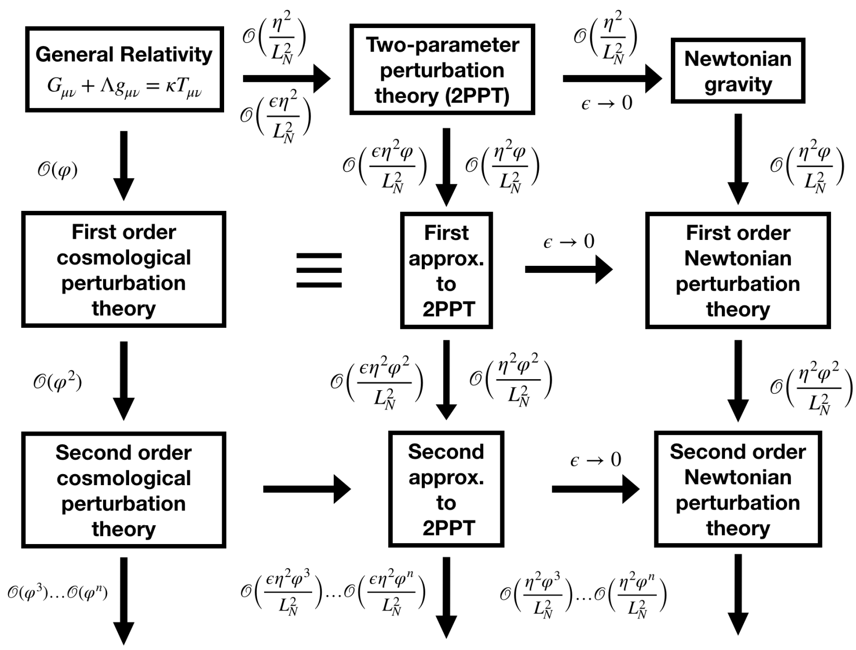


Figure 7.1.: A flowchart illustrating the differences and relationships between between CPT and 2PPT.

Similar expansions apply to all other variables in Eqs. (7.10)-(7.11), which can then be solved for order-by-order to get approximate solutions in the nonlinear regime of structure formation. The series solutions for Newtonian quantites can then be used to solve Eqs. (7.12), (7.14) and (7.25), where cosmological quantities are similarly expanded in $\varphi$, and the equations are again solved order-by-order in $\varphi$.

This further expansion should formally be regarded as a third (and separate) expansion to the two that have already been performed in 2PPT, this time associated with the linear fluctuations $\varphi$. In what follows, we will refer to terms of order $\sim \varphi^n$ as the "*n-th approximations*" to whatever equations they are intended to solve (e.g. the "$2^{\text{nd}}$ approximation to the 2PPT evolution equation" or the "$1^{\text{st}}$ approximation to the Newtonian Euler equation"). It is important to note that taking derivatives will not alter the powers of quantities in $\varphi$, as the series expansion associated with $\varphi$ does not require making any assumptions about spatial or temporal scales.

The corresponding cosmological expansions are

$$\delta = \delta^{(1)} + \frac{1}{2}\delta^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\delta^{(n)}}{n!} \tag{7.29}$$

$$\theta = \theta^{(1)} + \frac{1}{2}\theta^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\theta^{(n)}}{n!} \tag{7.30}$$

$$\psi = \psi^{(1)} + \frac{1}{2}\psi^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\psi^{(n)}}{n!} \tag{7.31}$$

$$\phi = \phi^{(1)} + \frac{1}{2}\phi^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\phi^{(n)}}{n!} \, . \tag{7.32}$$

For example, consider the following action of a spatial derivative on a product:

$$\phi^{(1)}U^{(1)} \sim \eta^4 \varphi^2 \, , \tag{7.33}$$

$$(\partial^i \phi^{(1)})U^{(1)} \sim \frac{\eta^5 \varphi^2}{L_{\mathrm{N}}} \, . \tag{7.34}$$

The action of the spatial derivative has introduced an additional *dimensionful* factor of $\frac{\eta}{L_{\mathrm{N}}}$; however, the term's power in $\varphi$ remains unchanged. In this sense, the expansion in the linear initial gravitational potential is more akin to a traditional perturbative expansion. In this paper, we proceed to solve the $\mathcal{O}\left(\frac{\eta^4}{L_{\mathrm{N}}^2}\right)$ field equations, by constructing a series of order by order approximations in $\varphi$. That is to say, our first approximation to the full $\mathcal{O}\left(\frac{\eta^4}{L_{\mathrm{N}}^2}\right)$ solution is given by solving the $\mathcal{O}\left(\frac{\eta^4\varphi}{L_{\mathrm{N}}^2}\right)$ field equations, using those $\mathcal{O}\left(\frac{\eta^4\varphi}{L_{\mathrm{N}}^2}\right)$ solutions to construct the quadratic source terms for the $\mathcal{O}\left(\frac{\eta^4\varphi^2}{L_{\mathrm{N}}^2}\right)$ field equations, then constructing $\mathcal{O}\left(\frac{\eta^4\varphi^3}{L_{\mathrm{N}}^2}\right)$ and $\mathcal{O}\left(\frac{\eta^4\varphi^4}{L_{\mathrm{N}}^2}\right)$ source terms, field equations and solutions, and so on and so forth. In this way, we recover the usual mathematical machinery of regular perturbation theories, illustrated in Equations (7.16), (7.18) and (7.19), namely solving a linear system first in the homogeneous case, then in the inhomogeneous case, for a progressively higher order series of source terms.

This can be seen easily by considering the fate of the terms contained within $\hat{D}\boldsymbol{u}_{\eta^4}$ in Equation (7.20) when expanded in this fashion. The lowest order an expanded term originating from this coupling can have is $\mathcal{O}\left(\frac{\eta^4\varphi^2}{L_{\mathrm{N}}^2}\right)$ - thus ensuring that we recover the *same* first order perturbation theory described by Equation (7.16) as a first approximation, given by the $\mathcal{O}\left(\frac{\eta^4\varphi}{L_{\mathrm{N}}^2}\right)$ field equations. Then, because the leading order Newtonian solutions are known to all orders, and the $\mathcal{O}\left(\frac{\eta^4\varphi}{L_{\mathrm{N}}^2}\right)$ is also known, the offending term just enters in the $\mathcal{O}(\varphi^2)$ equations as a regular source term as in normal cosmological perturbation theory. This implies that successive approximations to the full $\mathcal{O}\left(\frac{\eta^4}{L_{\rm N}^2}\right)$ dynamics can be solved for using the same methods as regular cosmological perturbation theory, the difference being solely that *different* inhomogeneous source terms will appear on the right hand side of the matrix equation. We also advise the reader to take note of the fact that if one were to retain tensor and vector degrees of freedom and then subject them to this procedure, the same results would hold. In this case the first approximation to the vector and tensor dynamics would decouple, allowing one to choose initial conditions with no vectors or tensors. The earliest approximation to the vector and tensor field equations that would have non-zero source terms would be the $\mathcal{O}(\varphi^2)$ approximation (e.g the vector constraint equation at $\mathcal{O}\left(\frac{\eta^3\varphi^2}{L_{\rm N}^2}\right)$ or the tensor evolution equation at $\mathcal{O}\left(\frac{\eta^4\varphi^2}{L_{\rm N}^2}\right)$). This would allow one to calculate induced vectors and tensors in the analogous way that they are calculated in second-order perturbation theory [88].

One way to conceptualise this process is to envisage that the full 2PPT field equations describe long-wavelength physics in a background universe that has nonlinear Newtonian density perturbations on short scales. This is an approximation to the full Einstein equations, where the nonlinearity is restricted to be only occurring in the short-wavelength regime. We then choose to think of this additional expansion as outlining a method to obtain approximate solutions to the 2P field equations, which are themselves approximations to the full Einstein equations.

The fundamental differences and relationships between the approach we propose, and the standard CPT and NPT approaches, are illustrated in Figure (7.1). At the top-left of this diagram we have the full, unperturbed theory of general relativity. CPT can be found within the solutions of the full general theory by hypothesising the existence of a background FLRW solution, and then considering perturbations around this background that are expanded in powers of the initial fluctuations, $\varphi$. First, second and higher-order CPT is obtained by simply working to higher and higher orders in $\varphi$, which corresponds to moving down the figure. On the other hand, the Newtonian limit of general relativity is obtained by performing an expansion in $\eta$. This is illustrated in the figure by first performing the 2PPT expansion (which corresponds to the first step right), and then taking the limit $\epsilon \to 0$ (which corresponds to the second step right). Both 2PPT and Newtonian theories can then also be expanded in $\varphi$, which again corresponds to moving down the figure.

The figure makes clear that 2PPT contains Newtonian gravity as a limiting case, but that it also provides a set of equations that describe first order cosmological perturbations in a universe with nonlinear structure. We have illustrated that 2PPT naturally contains Newtonian gravity within its structure by retaining both the $\mathcal{O}(\frac{\eta^2}{L_{\rm N}^2})$ and $\mathcal{O}(\frac{\epsilon\eta^2}{L_{\rm N}^2})$ labels on the left and right hand side of the 2PPT pathway. We can approximate the full 2PPT dynamics by constructing the $n^{\rm th}$ approximations in $\varphi$ to the subleading order 2PPT field equations in exactly the same way that we solve the NPT equations, understanding that our final solution will approximate the full 2PPT dynamics, which themselves are only an approximation to full general relativity. Newtonian expressions can be obtained from 2PPT by simply setting the cosmological parameter $\epsilon = 0$. This relationship is depicted in Figure (7.1); arrows denote approximations, so each node of the diagram can be considered as an approximation to all the previous nodes connected to it, and all successive nodes connected to a node are themselves approximations of that node.

We will demonstrate subsequently that first-order CPT and the first approximation to 2PPT are identical, and that the $\epsilon \to 0$ limit of 2PPT reproduces the first-order results of NPT. At second approximation, however, there is no longer an exact correspondence between CPT and 2PPT; The constraint equations differ, and the resulting phenomenology is therefore no longer the same. This will be explained in more detail in the sections that follow, followed by some discussion of exactly what effects are being included or neglected in each approach, as well as the corresponding benefits and drawbacks.

Whilst the equations for cosmological perturbations presented above are thought to be valid even in a spacetime with short-wavelength density fluctuations of $\mathcal{O}(1)$, this approximation narrows the regime of applicability to one at least in which it is possible to linearise the Newtonian field equations as a first approximation. We are therefore sacrificing some of the power of the 2PPT formalism; however, this method allows us to proceed analytically and therefore is valuable, at the very least as a consistency check for the two-parameter formalism, and more generally as it may highlight some of the key physics involved. It should however be emphasised that in general, one does *not* expect in general that a linearisation of the 2P field equations will give precisely the same result as cosmological perturbation theory (a direct linearisation of the Einstein field equation). This is because the 2PPT formalism contains extra restrictions on the size of small scale peculiar velocities relative to gravitational potentials and density perturbations, as is common in post-Newtonian gravity, and these restrictions are absent in cosmological perturbation theory.

## 7.4. Solutions to two-parameter perturbation theory

In Section 7.3, we outlined the utility of making further approximations to the 2PPT equations in order to find analytic solutions. In effect, this will involve solving the Newtonian equations perturbatively, and then considering the knock-on effect on the cosmological quantities (a type of cosmological back-reaction, from nonlinear structures on to the large-scale perturbations). In this section we will use and develop the techniques from Sections 4.1 and 4.2 to find explicit solutions to the 2PPT Eqs. (7.12)-(7.15).

This is achieved by inserting the perturbative expansions, as given in Eq. (7.26), into Eqs. (7.12)-(7.15). A first approximation to the 2PPT dynamics is made by neglecting any quadratic terms in $\varphi$, leading to a homogeneous set of PDEs that can be solved to find the first approximation to the 2PPT solutions. This is then followed in standard fashion, by calculating second approximations to 2PPT solutions using quadratic products of the first approximations as inhomogeneous source terms, continuing *ad infinitum* to higher orders.

In this section we will start by finding the first approximation to the 2PPT equations, before moving on to consider the scalar constraints at second approximation. We will then solve the relevant evolution equations, and discuss suitable initial conditions. These results will then all be used to calculate the matter bispectrum, which will be compared to the corresponding quantity in NPT and CPT.

### 7.4.1. First approximation to 2PPT

Linearising in $\varphi$, we find that the resulting system takes the following form:

$$\frac{1}{3}\nabla^2\phi^{(1)} + \mathcal{H}(\phi^{(1)\prime} + \psi^{(1)\prime} + 2U^{(1)\prime}) + (\psi^{(1)\prime\prime} + U^{(1)\prime\prime}) + 2\mathcal{H}'(\phi^{(1)} + U^{(1)}) = \frac{\mathcal{H}^2}{2}\delta^{(1)} , \tag{7.35}$$

$$\frac{1}{3}\nabla^2\psi^{(1)} - \mathcal{H}(\psi^{(1)\prime} + U^{(1)\prime}) - \mathcal{H}^2(\phi^{(1)} + U^{(1)}) = \frac{\mathcal{H}^2}{2}\delta^{(1)} , \tag{7.36}$$

as well as

$$\nabla^2\big(\psi^{(1)\prime} + \mathcal{H}\phi^{(1)}\big) = -\frac{3\mathcal{H}^2}{2}\theta^{(1)} , \tag{7.37}$$

$$\nabla^4(\phi^{(1)} - \psi^{(1)}) = 0 . \tag{7.38}$$

These equations take a form familiar from linear CPT in conformal Newtonian gauge. Using Eq. (7.38) to eliminate $\phi^{(1)}$ for $\psi^{(1)}$, and then subtracting Eq. (7.36) from Eq. (7.35), we are left with the analogous evolution equation for the linearised scalar degree of freedom:

$$\left(\psi^{(1)\prime\prime} + U^{(1)\prime\prime}\right) + 3\mathcal{H}\left(\psi^{(1)\prime} + U^{(1)\prime}\right) = 0 \,. \tag{7.39}$$

Although we have already technically established that

$$U^{(1)} = \text{ const } \implies U^{(1)\prime} = 0 \tag{7.40}$$

in the EdS case via our study of Newtonian perturbation theory, the form of this equation prompts a number of questions. In particular, is it consistent to consider $U^{(1)}$ and $\psi^{(1)}$ to be entirely separate degrees of freedom in this setup? How should we interpret $\psi^{(1)}$? How should we connect the initial conditions, $\varphi(x)$, defined as a continuous function on all length scales, to $U^{(1)}$ and $\phi^{(1)}$, when each of them are defined only on the spatial scales where the expansion used to derive them applies?

To answer these questions, we take note of the fact that since Eqs. (7.35) and (7.36) are linear, one can always write the solutions in the form

$$\psi^{(1)} = \psi^{(1)R} + \psi^{(1)N} \,, \tag{7.41}$$

where $\psi^{(1)N}$ satisfies the Newton-Poisson equation *on large scales*,

$$\nabla^2\psi^{(1)N} = \frac{3\mathcal{H}^2}{2}\delta^{(1)N} \,, \tag{7.42}$$

and where $\psi^{(1)R}$ is whatever is left (the relativistic contribution). The large-scale density contrast and velocity can also be split in a corresponding fashion:

$$\delta^{(1)} = \delta^{(1)R} + \delta^{(1)N} \,, \tag{7.43}$$

$$\theta^{(1)} = \theta^{(1)R} + \theta^{(1)N} \,. \tag{7.44}$$

We can then regard $\psi^{(1)N}$, $\delta^{(1)N}$ and $\theta^{(1)N}$ to be the long-wavelength extension of the quantities $U^{(1)}$, $\delta^{(1)}_{\mathrm{N}}$ and $\theta^{(1)}_{\mathrm{N}}$.

The evolution equation can then be written as

$$\begin{aligned}&\left(\psi^{(1)R\prime\prime}+\psi^{(1)N\prime\prime}+U^{(1)\prime\prime}\right)\\ +3\mathcal{H}&\left(\psi^{(1)R\prime}+\psi^{(1)N\prime}+U^{(1)\prime}\right)=0\ .\end{aligned} \tag{7.45}$$

Since the Newtonian and cosmological perturbation theories have identical gravitational potentials on all scales at first-order, this motivates us to choose our initial conditions as

$$\psi^{(1)N}+U^{(1)}=\varphi\ , \tag{7.46}$$

where $\varphi$ now has support on all spatial scales, making the extension of the solution to large scales explicit. We can then consistently also choose $\psi^{(1)R}=0$ at all times, which is equivalent to the statement that there is no leading-order large-scale correction to the Newtonian gravitational potential.

Whilst this discussion makes explicit the extension of the Newtonian solution to all scales, it is a notational inconvenience to persevere with so many different terms, especially when many of these terms always appear together alongside each other in our equations. We will therefore implement the following re-labellings:

$$U^{(1)}+\psi^{(1)N}\rightarrow U^{(1)}=\varphi\ , \tag{7.47}$$

$$\psi^{(1)R}\rightarrow\psi^{(1)}=0\ , \tag{7.48}$$

$$\delta^{(1)N}+\delta^{(1)}_{\mathrm{N}}\rightarrow\delta^{(1)}_{\mathrm{N}}=\frac{2\nabla^2\varphi}{3\mathcal{H}^2}\ , \tag{7.49}$$

$$\delta^{(1)R}\rightarrow\delta^{(1)}\ , \tag{7.50}$$

$$\theta^{(1)N}+\theta^{(1)}_{\mathrm{N}}\rightarrow\theta^{(1)}_{\mathrm{N}}\ , \tag{7.51}$$

$$\theta^{(1)R}\rightarrow\theta^{(1)}\ . \tag{7.52}$$

This simply allows us to recharacterize $\psi^{(1)}$, $\delta^{(1)}$ and $\theta^{(1)}$ as being the purely relativistic corrections to the all-scales Newtonian quantities $U^{(1)}$, $\delta^{(1)}_{\mathrm{N}}$ and $\theta^{(1)}_{\mathrm{N}}$, without having to introduce any superfluous degrees of freedom.

Having done this, we can then use the field equations to work out the rest of the cosmological quantities. In particular, Eq. (7.36) yields

$$\mathcal{H}^2(U^{(1)})-\frac{\mathcal{H}^2}{2}(\delta^{(1)})=0 \tag{7.53}$$

$$\implies\ \delta^{(1)}=-2U^{(1)}=-2\varphi\ , \tag{7.54}$$

which we recognise as the gauge correction to the dark matter overdensity in Poisson gauge [112], while Eq. (7.37) guarantees

$$\theta^{(1)} = 0 \ . \tag{7.55}$$

The significance of this result is immediately apparent; if you linearise the short scale nonlinear structures in the two-parameter perturbation theory equations, you simply obtain the results for the relativistic corrections in standard first-order CPT in Poisson gauge. This is not at all surprising, given that the linear terms in the field equations satisfy equations of the same form [52]. We will now proceed to the second approximation.

### 7.4.2. Second approximation to the scalar constraints

To obtain the second approximation to the combination $\psi^{(2)} - \phi^{(2)}$, it is necessary to consider the application of the operator $\partial_i \partial^j$ to Eq. (7.13). As described in Section 7.3, this will mean that we have to consider terms that appear up to order $\sim \eta^6/L_{\mathrm{N}}^4$ in order to capture the full dynamics, due to the possible change in size of terms when using the "inverse Laplacian" operator on products. As is also discussed in Ref. [127], it is necessary to include any terms that may not be included in Eq. (7.13), but that may nonetheless end up contributing to the $\mathcal{O}(\eta^4/L_{\mathrm{N}}^4)$, $\mathcal{O}(\eta^5/L_{\mathrm{N}}^4)$ or $\mathcal{O}(\eta^6/L_{\mathrm{N}}^4)$ expressions that result from the application of the $\partial_i \partial^j$ operator to (7.13).

For example, we should consider those terms contained within $R^i{}_j \sim \frac{\eta^5}{L_{\mathrm{N}}^2}$ that remain $\mathcal{O}\left(\frac{\eta^5}{L_{\mathrm{N}}^4}\right)$ or $\mathcal{O}\left(\frac{\eta^6}{L_{\mathrm{N}}^4}\right)$ after the application of two spatial derivatives, but not those that would be promoted to $\mathcal{O}\left(\frac{\eta^7}{L_{\mathrm{N}}^2}\right)$ or higher order.

For our present purposes, we need to calculate $\dfrac{1}{3}\nabla^4(\psi^{(2)} - \phi^{(2)})$ up to $\mathcal{O}(\eta^6/L_{\mathrm{N}}^4)$. This is required to perform the our study consistently, but it will also enable us to discuss the pros and cons of this approach, and highlight the areas where differences and benefits can occur between this formalism and standard CPT. Schematically, we can write

$$\frac{1}{3}\nabla^4(\psi^{(2)} - \phi^{(2)}) = \mathcal{S}_4^{(2)} + \mathcal{S}_5^{(2)} + \mathcal{S}_6^{(2)} \ , \tag{7.56}$$

where

$$
\begin{aligned}
\mathcal{S}_4^{(2)} &= 16\pi a^2\ \bar{\rho}\ v_{\mathrm{N}i}^{(1)}\ \partial^i\theta_{\mathrm{N}}^{(1)} + 8\pi a^2\bar{\rho}(\theta_{\mathrm{N}}^{(1)})^2 + 8\pi a^2\bar{\rho}\ \partial_i v_{\mathrm{N}j}^{(1)}\ \partial^j v_{\mathrm{N}}^{(1)i} \\
&\quad - \frac{2}{3}(\nabla^2 U^{(1)})^2 - \frac{14}{3}\partial_i\partial_j U^{(1)}\partial^i\partial^j U^{(1)} \\
&\quad - \frac{16\pi a^2\bar{\rho}}{3}\ \partial_j v_{\mathrm{N}i}^{(1)}\ \partial^j v_{\mathrm{N}}^{(1)i} - \frac{16\pi a^2\bar{\rho}}{3}\ v_{\mathrm{N}i}^{(1)}\ \nabla^2 v_{\mathrm{N}}^{(1)i} \\
&\quad - 8\partial_i\nabla^2 U^{(1)}\partial^i U^{(1)} - \frac{8}{3}U^{(1)}\nabla^4 U^{(1)} - \frac{8}{3}\psi^{(1)}\nabla^4 U^{(1)} \sim \mathcal{O}\big(\frac{\eta^4}{L_{\mathrm{N}}^4}\big)\ , \qquad (7.57)
\end{aligned}
$$

$$
\mathcal{S}_5^{(2)} = 16\pi a^2\ \bar{\rho}\ v_{\mathrm{N}i}^{(1)}\ \partial^i\theta^{(1)} - 8\partial_i\nabla^2 U^{(1)}\partial^i\psi^{(1)} - \frac{16\pi a^2\bar{\rho}}{3}\ v_i^{(1)}\ \nabla^2 v_{\mathrm{N}}^{(1)i} \sim \mathcal{O}\big(\frac{\eta^5}{L_{\mathrm{N}}^4}\big)\ , \qquad (7.58)
$$

$$
\begin{aligned}
\mathcal{S}_6^{(2)} &= 8\pi a^2\bar{\rho}\ \theta_{\mathrm{N}}^{(1)}\ \theta^{(1)} + 8\pi a^2\bar{\rho}\ \partial_i v_{\mathrm{N}j}^{(1)}\ \partial^j v^{(1)i} - \frac{2}{3}\nabla^2 U^{(1)}\nabla^2\psi^{(1)} \\
&\quad - \frac{14}{3}\partial_i\partial_j U^{(1)}\partial^i\partial^j\psi^{(1)} - \frac{16\pi a^2\bar{\rho}}{3}\ \partial_j v_{\mathrm{N}i}^{(1)}\ \partial^j v^{(1)i} \sim \mathcal{O}\big(\frac{\eta^6}{L_{\mathrm{N}}^4}\big)\ , \qquad (7.59)
\end{aligned}
$$

and where subscript $n$ indicates that the order of the quantity is $\sim \eta^n/L_{\mathrm{N}}^4$.

If we want to obtain a second approximation to the dynamics, we should take note of the following facts that considerably simplify the results of this calculation.

(i) The first-order large-scale relativistic corrections are all zero, apart from the density contrast $\delta^{(1)} = -2\varphi$, which receives a linear correction.

(ii) The density contrast only appears in a cubic product in Eq. (7.13).

(iii) Therefore, the only source terms that will contribute to the second approximation will be quadratic products of the Newtonian leading-order quantities.

(iv) Quadratic products of Newtonian quantities can be at maximum $\sim \eta^4$.

It is therefore only necessary to consider the terms at $\mathcal{O}(\eta^4/L_{\mathrm{N}}^4)$ to find the second approximation to this equation.

The reader should note that the second approximation to cosmological quantities will obey *different* equations to the second approximation to Newtonian quantities; we therefore expect $\psi^{(2)} \neq 0$, along with the rest of the second approximations to cosmological large-scale perturbations, even though the source terms are all Newtonian. Thus, when we come to consider the third approximation to Eq. (7.13), there will be contributions from terms like $\psi^{(2)}\nabla^4 U^{(1)}$, a coupling between an explicitly relativistic source term and a Newtonian one. Although the calculation is long, a third approximation to the dynamics of the two-parameter field equations is vastly preferable to full third-order CPT.

One benefit of the two-parameter approach is that the separation of the source terms allows us to see directly where Newtonian sources can be used, and where it is important to include the full relativistic solutions in the source terms.

Applying the logic presented above, we immediately see that $\mathcal{S}_5^{(2)} = 0$ and $\mathcal{S}_6^{(2)} = 0$ (although we note that $\mathcal{S}_5^{(3)} \neq 0$ and $\mathcal{S}_6^{(3)} \neq 0$, due to the fact that $\psi^{(2)} \neq 0$). We are therefore left with

$$\begin{aligned}\frac{1}{3}\nabla^4(\psi^{(2)} - \phi^{(2)}) = {} & 16\pi a^2\, \bar\rho\, v^{(1)}_{\mathrm{N}i}\, \partial^i\theta^{(1)}_{\mathrm{N}} + 8\pi a^2\bar\rho(\theta^{(1)}_{\mathrm{N}})^2 + 8\pi a^2\bar\rho\, \partial_i v^{(1)}_{\mathrm{N}j}\, \partial^j v^{(1)i}_{\mathrm{N}} \\ & - \frac{2}{3}(\nabla^2 U^{(1)})^2 - \frac{14}{3}\Big(\partial_i\partial_j U^{(1)}\Big)\Big(\partial^i\partial^j U^{(1)}\Big) \\ & - \frac{16\pi a^2\bar\rho}{3}\, v^{(1)}_{\mathrm{N}i}\, \nabla^2 v^{(1)i}_{\mathrm{N}} - 8\Big(\partial_i\nabla^2 U^{(1)}\Big)\partial^i U^{(1)} \\ & - \frac{16\pi a^2\bar\rho}{3}\, \partial_j v^{(1)}_{\mathrm{N}i}\, \partial^j v^{(1)i}_{\mathrm{N}} - \frac{8}{3}U^{(1)}\nabla^4 U^{(1)} \sim \mathcal{O}\big(\frac{\eta^4}{L^4_{\mathrm{N}}}\big)\,. \end{aligned} \tag{7.60}$$

We can now directly insert our solutions, given here in terms of the initial gravitational potential fluctuation, $\varphi$;

$$U^{(1)} = \varphi\,, \tag{7.61}$$

$$\delta^{(1)}_{\mathrm{N}} = \frac{2\nabla^2\varphi}{3\mathcal{H}^2}\,, \tag{7.62}$$

$$\theta_{\mathrm{N}} = \frac{-2\nabla^2\varphi}{3\mathcal{H}} \qquad \Longrightarrow \qquad v_{\mathrm{N}i} = \frac{-2\partial_i\varphi}{3\mathcal{H}}\,, \tag{7.63}$$

$$v^{(1)}_{\mathrm{N}} = \frac{-2\varphi}{3\mathcal{H}}\,. \tag{7.64}$$

Evaluating this, and using the identity $\nabla^4(\varphi^2) = 2\nabla^2\varphi\nabla^2\varphi + \varphi\nabla^4\varphi + 4\partial_i\partial_j\varphi\partial^i\partial^j\varphi + 8\partial^j\nabla^2\partial_j\varphi$, we establish

$$\psi^{(2)} - \phi^{(2)} = -4\varphi^2 - \frac{10}{3}\nabla^{-4}\Big[\nabla^2(\nabla\varphi)^2 - 3\partial_i\partial^j\big(\partial^i\varphi\partial_j\varphi\big)\Big]\,. \tag{7.65}$$

This is identical to the constraint calculated on the second order gravitational slip, $\Psi_2 - \Phi_2$ in Chapter 4. In Chapter 4, this constraint was calculated for the $\Lambda$CDM case (Equation (4.88)), but the Einstein-de Sitter limit can be easily shown to be identical to Equation (7.65). We could have anticipated this by considering Equation (7.13), directly neglecting the cubic term $\delta v^i_{\mathrm{N}} v_{\mathrm{N}j}$, and then performing exactly the same calculation as is done in standard second-order cosmological perturbation theory [89]. However doing the calculation this way misses the possibility of any interactions between linear relativistic corrections and linear Newtonian quantities. In particular terms in $\mathcal{S}_5^{(2)}$ and $\mathcal{S}_6^{(2)}$ can contribute to the quantity $\nabla^4(\phi^{(2)} - \phi^{(2)}) \sim \mathcal{O}\big(\frac{\eta^6}{L_{\rm N}^4}\big)$, and performing the calculation as is done in second-order perturbation theory misses this possibility. This is a direct result of the fact that there are no relativistic corrections to any first-order quantities, apart from the density contrast.

The third approximation of this problem (although still involved) is *considerably* easier than calculating results in full third-order cosmological perturbation theory, and it is easy to see that there will be interactions between quantities like $\psi^{(2)}$ and $U^{(1)}$ that explicitly demonstrate couplings between long-wavelength relativistic corrections and the linear Newtonian potential. Although such terms also arise naturally in third-order CPT (alongside many other such terms that are neglected in this scheme), in a realistic universe we expect the terms that arise in the third approximation to 2PPT to be the largest and most relevant ones, as the full two-parameter perturbation theory equations are valid even in universes with highly nonlinear structures on short scales.

### 7.4.3. Evolution of gravitational potentials in 2PPT

The second approximation to the 2PPT evolution equation can be written as

$$\begin{aligned}\Big(\frac{1}{2}\psi^{(2)} + \frac{1}{2}U^{(2)}\Big)'' + 3\mathcal{H}\Big(\frac{1}{2}\psi^{(2)} + \frac{1}{2}U^{(2)}\Big)' = \; & \frac{4\pi a^2\bar{\rho}}{3}(v_{\rm N}^{(1)})^2 + \mathcal{H}\Big(\frac{1}{2}\psi^{(2)\prime} - \frac{1}{2}\phi^{(2)\prime}\Big) \\ & + \frac{1}{6}\nabla^2(\psi^{(2)} - \phi^{(2)}) + \frac{7}{6}(\nabla U^{(1)})^2 \\ & + \frac{2}{3}(\phi^{(1)} + \psi^{(1)} + 2U^{(1)})\nabla^2 U^{(1)} \,. \end{aligned} \tag{7.66}$$

Inserting our first approximations, and taking note of the fact that $\psi^{(2)\prime} - \phi^{(2)\prime} = 0$ in Einstein-de Sitter, we recover

$$\begin{aligned}(U^{(2)} + \psi^{(2)})'' + 3\mathcal{H}(U^{(2)} + \psi^{(2)})' = \; & \frac{8\pi a^2\bar{\rho}}{3}(v_{\rm N}^{(1)})^2 + \frac{1}{3}\nabla^2(\psi^{(2)} - \phi^{(2)}) \\ & + \frac{7}{3}(\nabla\varphi)^2 + \frac{8}{3}\varphi\nabla^2\varphi \,, \\ = \; & \frac{10}{3}\nabla^{-2}\partial_i\partial^j\left(\partial^i\varphi\partial_j\varphi\right) - (\nabla\varphi)^2 \,, \end{aligned} \tag{7.67}$$

which is an inhomogeneous evolution equation for $(U^{(2)} + \psi^{(2)})$, and which (given that the relationship between $\psi^{(2)}$ and $\phi^{(2)}$ is the same as in CPT) is of precisely the same form as the second-order CPT evolution equation (4.89).

Without assuming anything about $U^{(2)}$, we can solve directly for the combination

$(U^{(2)} + \psi^{(2)})$, yielding the solution

$$(U^{(2)} + \psi^{(2)}) = (U^{(2)} + \psi^{(2)})_0 + (U^{(2)} + \psi^{(2)})_P \;, \tag{7.68}$$

where $(U^{(2)} + \psi^{(2)})_P$ is the particular solution, found via the same method detailed in Section 4.2.3,

$$(U^{(2)} + \psi^{(2)})_P = \frac{a(\tau)}{14}\left[\frac{10}{3}\nabla^{-2}\partial_i\partial^j\left(\partial^i\varphi\partial_j\varphi\right) - (\nabla\varphi)^2\right] \;, \tag{7.69}$$

where $\psi_0^{(2)}$ is an initial condition. We can now use the fact that the leading-order 2PPT field equations for $U^{(2)}$ are identical NPT to find a separate expression for $U^{(2)}$.

Performing a Fourier transform, and using the identity $\nabla^2\left(\partial^i\varphi\partial_i\varphi\right) = 2\partial^i\partial^j\varphi\;\partial_i\partial_j\varphi + 2\nabla^2\partial_i\varphi\partial^i\varphi$, it is possible to show that the RHS of Eq. (7.69) is, in fact, precisely equal to $U^{(2)}$ (as calculated in NPT, using the second-order Newton-Poisson equation to relate $\delta_{\mathrm{N}}^{(2)}(k)$ to $U^{(2)}(k)$). We are therefore left with

$$U^{(2)} = a(\tau)\left[\frac{1}{6}(\nabla\varphi)^2 - \frac{10}{21}\boldsymbol{\Psi}_0\right] , \qquad \psi^{(2)} = \psi_0^{(2)} \;, \qquad \text{and} \qquad \phi^{(2)} = \phi_0^{(2)} \;, \tag{7.70}$$

where $\phi_0^{(2)}$ can be obtained from $\psi_0^{(2)}$ using the constraint from Equation (7.65), and $\boldsymbol{\Psi}_0$ is the quantity defined in Eq. (4.122), the *Newtonian kernel*. This result demonstrates that purely relativistic effects only arise as a result of second-order initial conditions in the second approximation to 2PPT in Einstein-de Sitter universes, as they do in second-order perturbation theory [112]. The same cannot be said for the 2PPT treatment in ΛCDM however - we will examine this issue in much more detail in Chapter 8.

The second approximation to the remaining 2PPT constraint equation,

$$\begin{aligned}\frac{1}{3}\nabla^2\psi_0^{(2)} - \mathcal{H}(\psi_0^{(2)\prime} + U^{(2)\prime}) - \mathcal{H}^2(\phi_0^{(2)} + U^{(2)}) = \frac{4\pi a^2\bar{\rho}}{3}\delta^{(2)} + \frac{8\pi a^2\bar{\rho}}{3}(v_{\mathrm{N}}^{(1)})^2 \\ - (\nabla\varphi)^2 - \frac{8}{3}\varphi\nabla^2\varphi \;, \end{aligned} \tag{7.71}$$

allows us to write the second approximation to the density contrast, $\delta^{(2)}$, in terms of the initial conditions to the potentials $\psi_0^{(2)}$ and $\phi_0^{(2)}$ as follows:

$$\delta^{(2)} = 2\left(\frac{1}{3\mathcal{H}^2}\nabla^2\psi_0^{(2)} - \phi_0^{(2)}\right) + \frac{10}{9\mathcal{H}^2}(\nabla\varphi)^2 + \frac{16}{3\mathcal{H}^2}\varphi\nabla^2\varphi - 4a(\tau)\left[\frac{1}{6}(\nabla\varphi)^2 - \frac{10}{21}\boldsymbol{\Psi}_0\right] . \tag{7.72}$$

Having identified the importance of second-order initial conditions, let us now turn to how these should be calculated.

### 7.4.4. Initial conditions in 2PPT

In standard cosmological perturbation theory, initial conditions for the growth of structure are usually specified using the curvature perturbation on uniform density hypersurfaces, $\zeta$, which can be connected to the output of various theories of the early Universe (e.g. inflationary models). Different models lead to different parameterisations of the second-order curvature perturbation in terms of the first, which can be written as $\zeta^{(2)} = 2a_{\rm NL}\zeta^{(1)2}$. One can also calculate $\zeta^{(2)}$ directly (see for example [31]), which using the Einstein equations and energy conservation equation gives

$$\phi_0^{(2)} = -\frac{3}{5}\zeta^{(2)} + \frac{16}{3}\varphi^2 + 2\nabla^{-4}\left[\nabla^2(\nabla\varphi)^2 - 3\partial_i\partial^j\left(\partial^i\varphi\partial_j\varphi\right)\right]. \tag{7.73}$$

However, when working with 2PPT we must take note of the fact that the second approximation to the 2PPT equations do not have precisely the same structure as the second-order CPT equations. In particular, the 00-field equation in second-order CPT contains the quadratic source term $-4\mathcal{H}^2\varphi^2$, which is absent in the second approximation to the corresponding 2PPT equation (due to it being of order $\sim \eta^6/L_{\rm N}^2$). We must therefore take some care in interpreting quantities like $\psi_0^{(2)}$ and $\phi_0^{(2)}$, as having a different 00-field equation implies that Eq. (7.73) is no longer true, and correspondingly the initial conditions may have to be modified.

Our physical interpretation of solving the 2PPT equations in the way outlined in this paper is that it systematically highlights which higher-order terms from regular perturbation theory should be amplified by the presence of nonlinear structures at late times in the universe. This means that the second approximation to 2PPT will not contain all terms that appear in full second-order CPT, as certain terms were never present in the full 2PPT system to begin with. We suggest that these terms are the ones that are not prone to being amplified by the presence of nonlinear structures (at least, not ones that can be described using post-Newtonian expansions). As the linearly evolving parts of both the second-order CPT metric scalars and the 2PPT metric scalars are identical, and equal to the second-order Newtonian gravitational potential, the question of appropriate initial conditions for the 2PPT metric scalars would then appear to be most appropriately specified by simply using the initial conditions from second-order CPT. As second-order CPT should be accurate up until the formation of nonlinear structures, any terms that exist within the second-order initial conditions that are "too small" (in terms of the 2PPT counting scheme) will be sub-dominant to those that 2PPT identifies will be amplified by the presence of nonlinear structure.

Let us formalise this choice. We choose a moment in conformal time, $\tau_{\rm cross}$, which should be deep in the matter-dominated era, but before significant growth of non-linear structure, which we will refer to as the "crossover time". At that moment, we switch from using the second-order CPT equations to using the second approximation to the 2PPT equations, which formally allows for some traditional second-order terms to become larger. Our choice of initial conditions is automatically consistent with the second approximation to the 2PPT scalar constraint on $\psi^{(2)}$ and $\phi^{(2)}$, as that constraint is identical to the one in regular cosmological perturbation theory, and has the benefit of ensuring that the metric is continuous at the crossover time. Our choice can therefore be written as

$$\begin{aligned}\phi_0^{(2)}(\tau_{\rm cross}) = \Phi_{2in}(\tau_{\rm cross}) &= 2\Big[\varphi^2 + \nabla^{-4}\Big(\nabla^2(\nabla\varphi)^2 - 3\partial_i\partial^j\left(\partial^i\varphi\partial_j\varphi\right)\Big)\Big] \\ &= 2\varphi^2 + 12\,\mathbf{\Theta}_0\;, \end{aligned}\tag{7.74}$$

$$\begin{aligned}\psi_0^{(2)}(\tau_{\rm cross}) = \Psi_{2in}(\tau_{\rm cross}) &= 2\Big[-\varphi^2 - \frac{2}{3}\nabla^{-4}\Big(\nabla^2(\nabla\varphi)^2 - 3\partial_i\partial^j\left(\partial^i\varphi\partial_j\varphi\right)\Big)\Big] \\ &= -2\varphi^2 - 8\,\mathbf{\Theta}_0\;, \end{aligned}\tag{7.75}$$

where $\mathbf{\Theta}_0$ is the quantity defined in Eq. (4.87). This choice ensures that $g_{\mu\nu}(\tau_{\rm cross})^{\rm 2PPT} = g_{\mu\nu}(\tau_{\rm cross})^{\rm CPT}$, at the expense of the appearance of a negligible discontinuity in the second approximation to the 2PPT dark matter density contrast and peculiar velocity.

Directly calculating the second approximation to the 2PPT dark matter density contrast, using the second approximation to the 2PPT field equations, we obtain

$$\delta^{(2)} = -4\varphi^2 - 24\,\mathbf{\Theta}_0 + \left[-\frac{22}{9\mathcal{H}^2}(\nabla\varphi)^2 + \frac{8}{3\mathcal{H}^2}\varphi\nabla^2\varphi + \frac{16}{7\mathcal{H}^2}\mathbf{\Psi}_0\right]\,, \tag{7.76}$$

which is very similar (but not identical) to the relativistic correction to the second-order density contrast in regular CPT on an Einstein-de Sitter background. The difference arises due to the 00-field equation in the second approximation to 2PPT and second-order CPT taking different forms; Specifically, the term $-4\mathcal{H}^2\varphi^2$ in the regular second-order CPT 00-field equation is no longer present in the second approximation to 2PPT, which results in a net change in sign for the term $-4\varphi$ in Eq.

(7.76), as compared to the corresponding expression in second-order perturbation theory. We note that this may not be the only choice available for initial conditions in 2PPT, and that it may be possible that other choices could arise from repeating the calculation of second-order initial conditions performed in [94], but using the second approximation to the 2PPT equations instead of the full second-order Einstein equation. This may prove to be an interesting project for the future.

### 7.4.5. Statistics in 2PPT

We now arrive at the question of calculating statistics using 2PPT. We will use the intrinsic dark matter bispectrum as our example statistic, as it is one of easiest to calculate. One can write the expression for this as

$$
\begin{aligned}
&(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)B(k_1,k_2,k_3)\\
&\quad=\left\langle\left(\delta^{(0,2)}+\delta^{(1,0)}+\dots\right)\left(\delta^{(0,2)}+\delta^{(1,0)}+\dots\right)\left(\delta^{(0,2)}+\delta^{(1,0)}+\dots\right)\right\rangle\\
&\simeq\left\langle\left(\delta_{\mathrm{N}}+\delta\right)\left(\delta_{\mathrm{N}}+\delta\right)\left(\delta_{\mathrm{N}}+\delta\right)\right\rangle\\
&\simeq\left\langle\left(\delta_{\mathrm{N}}^{(1)}+\frac{1}{2}\delta_{\mathrm{N}}^{(2)}+\dots+\delta^{(1)}+\frac{1}{2}\delta^{(2)}+\dots\right)^3\right\rangle\,.
\end{aligned}
\tag{7.77}
$$

It is easy to see that at leading order this reduces to

$$
\begin{aligned}
&(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)B_{\mathrm{2PPT}}(k_1,k_2,k_3)\\
&\quad=(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\Big(B_{\mathrm{N}}(k_1,k_2,k_3)+B_{\mathrm{R}}(k_1,k_2,k_3)\Big)\,,
\end{aligned}
\tag{7.78}
$$

where $B_{\mathrm{N}}$ is the Newtonian bispectrum and $B_{\mathrm{R}}$ is a relativistic correction. $B_{\mathrm{N}}$ is given by

$$
(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)B_{\mathrm{N}}(k_1,k_2,k_3)=\frac{1}{2}\left\langle\delta_{\mathrm{N}}^{(1)}(\boldsymbol{k}_1)\delta_{\mathrm{N}}^{(1)}(\boldsymbol{k}_2)\delta_{\mathrm{N}}^{(2)}(\boldsymbol{k}_3)\right\rangle+\ \text{2 cycl. perms}\,,
\tag{7.79}
$$

and $B_R$ is the 2PPT relativistic bispectrum, given by

$$
\begin{aligned}
&(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)B_R(k_1,k_2,k_3)\\
=&\quad\left\langle\delta_{\mathrm{N}}^{(1)}(\boldsymbol{k}_1)\delta_{\mathrm{N}}^{(1)}(\boldsymbol{k}_2)\delta^{(2)}(\boldsymbol{k}_3)\right\rangle+\ \text{2 cycl. perms}\\
&+\left\langle\delta^{(1)}(\boldsymbol{k}_1)\delta_{\mathrm{N}}^{(1)}(\boldsymbol{k}_2)\delta_{\mathrm{N}}^{(2)}(\boldsymbol{k}_3)\right\rangle+\ \text{2 cycl. perms}\\
&+\left\langle\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(2)}(\boldsymbol{k}_3)\right\rangle+\ \text{2 cycl. perms}\,.
\end{aligned}
\tag{7.80}
$$

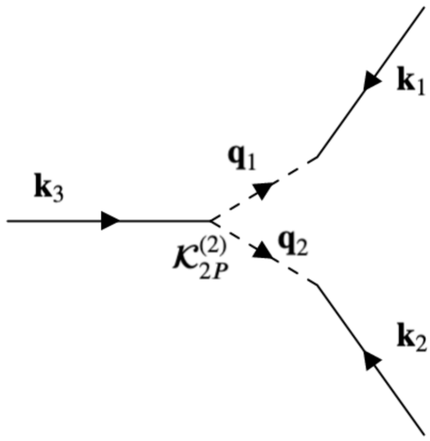


Figure 7.2.: One of the three diagrams for the tree level 2PPT bispectrum, $\left\langle \delta^{(1)}_{2p}(\boldsymbol{k}_1)\delta^{(1)}_{2p}(\boldsymbol{k}_2)\delta^{(2)}_{2p}(\boldsymbol{k}_3)\right\rangle_T$. The other diagrams are simply rotations of this one.

This approach has the benefit of directly revealing which terms in each statistic refer to a specific scale, and which terms arise from interactions between scales. It is, however, an extremely cumbersome way of performing the calculation; Instead let us define $\delta^{(n)}_{\mathrm{2PPT}} = \delta^{(n)}_{\mathrm{N}} + \delta^{(n)}$, and calculate

$$\begin{aligned}(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)B_{\mathrm{2PPT}}(k_1,k_2,k_3)\\ = \quad \left\langle \delta^{(1)}_{\mathrm{2PPT}}(\boldsymbol{k}_1)\delta^{(1)}_{\mathrm{2PPT}}(\boldsymbol{k}_2)\delta^{(2)}_{\mathrm{2PPT}}(\boldsymbol{k}_3)\right\rangle + \ \text{2 cycl. perms}\ ,\end{aligned} \tag{7.81}$$

where $\delta^{(n)}_{\mathrm{2PPT}} \equiv \delta^{(n)}_{\mathrm{N}} + \delta^{(n)}$. Doing this allows us to calculate the modified 2PPT kernel, defined implicitly by

$$\delta^{(2)}_{\mathrm{2PPT}}(k) = \int \frac{\mathrm{d}^3q_1\mathrm{d}^3q_2}{(2\pi)^3}\,\mathcal{K}^{(2)}_{\mathrm{2PPT}}(\boldsymbol{q}_1,\boldsymbol{q}_2,\tau)\;\delta^{(1)}_{\mathrm{2PPT}}(\boldsymbol{q}_1,\tau)\;\delta^{(1)}_{\mathrm{2PPT}}(\boldsymbol{q}_2,\tau)\ , \tag{7.82}$$

which can be used in a modified set of Feynman rules, where instead of using the relativistic kernel at each vertex we use $\mathcal{K}^{(2)}_{\mathrm{2PPT}}$. We can then just read off the following expression from Figure 4.12,

$$B_{\mathrm{2PPT}}(k_1,k_2,k_3) = \mathcal{K}^{(2)}_{\mathrm{2PPT}}(\boldsymbol{k}_1,\boldsymbol{k}_2)P_{\mathrm{2PPT}}(k_1)P_{\mathrm{2PPT}}(k_2) + \ \text{2 cycl. perms}\ldots\ , \tag{7.83}$$

where $P_{\rm 2PPT}(k)$ is the tree level two-parameter matter power spectrum, defined in the usual way by

$$(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)P_{\rm 2PPT}(k_1) = \left\langle \left(\delta_{\rm N}^{(1)}(\boldsymbol{k}_1)+\delta^{(1)}(\boldsymbol{k}_1)\right)\left(\delta_{\rm N}^{(1)}(\boldsymbol{k}_2)+\delta^{(1)}(\boldsymbol{k}_2)\right)\right\rangle . \quad (7.84)$$

All that remains is then to directly calculate the 2PPT kernel for the dark matter density contrast. Starting with Eq. (7.76), we take a Fourier transform to obtain

$$\begin{aligned}\delta_{\rm 2PPT}^{(2)}(k) = \int {\rm d}^3q_1\,{\rm d}^3q_2\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\,\varphi(\boldsymbol{q}_1)\,\varphi(\boldsymbol{q}_2)&\\ \times\Bigg[-4-\frac{4\,\boldsymbol{q}_1\cdot\boldsymbol{q}_2}{k^2}+\frac{22\,\boldsymbol{q}_1\cdot\boldsymbol{q}_2}{9\mathcal{H}^2}-\frac{4}{3\mathcal{H}^2}(q_1^2+q_2^2)&\\ +\frac{12}{k^4}\Big(q_1^2q_2^2+(q_1^2+q_2^2)(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)+(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2\Big)&\\ +\frac{8}{7\mathcal{H}^2k^2}\big(q_1^2q_2^2-(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2\big)&\\ +\frac{4}{9\mathcal{H}^4}\Big(\frac{10}{7}q_1^2q_2^2+(q_1^2+q_2^2)(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)+\frac{4}{7}(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2\Big)\Bigg] .&\end{aligned} \quad (7.85)$$

In order to get this expression into the required form, we can use

$$\delta_{\rm 2PPT}^{(1)} = \delta_{\rm N}^{(1)}+\delta^{(1)} = \frac{2\nabla^2\varphi}{3\mathcal{H}^2}-2\varphi\,, \quad (7.86)$$

which implies

$$\varphi(k) = -\frac{3\mathcal{H}^2}{2k^2}\left(1+\frac{3\mathcal{H}^2}{k^2}\right)^{-1}\delta_{\rm 2PPT}^{(1)}(k)\,, \quad (7.87)$$

to relate $\varphi(q_1)$ and $\varphi(q_2)$ to $\delta_{\rm 2PPT}^{(1)}(q_1)$ and $\delta_{\rm 2PPT}^{(1)}(q_2)$, yielding the final expression for the second-order 2PPT matter density kernel

$$\mathcal{K}^{(2)}_{\rm 2PPT}(\boldsymbol{q}_1,\boldsymbol{q}_2,k,\tau) = \frac{9\mathcal{H}^4}{4\,q_1^2q_2^2}\left(1+\frac{3\mathcal{H}^2}{q_1^2}\right)^{-1}\left(1+\frac{3\mathcal{H}^2}{q_2^2}\right)^{-1} \tag{7.88}$$

$$\times\Bigg[-4-\frac{4\,\boldsymbol{q}_1\cdot\boldsymbol{q}_2}{k^2}+\frac{22\,\boldsymbol{q}_1\cdot\boldsymbol{q}_2}{9\mathcal{H}^2}-\frac{4}{3\mathcal{H}^2}(q_1^2+q_2^2)$$
$$+\frac{12}{k^4}\Big(q_1^2q_2^2+(q_1^2+q_2^2)(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)+(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2\Big)$$
$$+\frac{8}{7\mathcal{H}^2k^2}\big(q_1^2q_2^2-(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2\big)$$
$$+\frac{4}{9\mathcal{H}^4}\Big(\frac{10}{7}q_1^2q_2^2+(q_1^2+q_2^2)(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)+\frac{4}{7}(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2\Big)\Bigg]\,. \tag{7.89}$$

This equation can be written more compactly as

$$\mathcal{K}^{(2)}_{\rm 2PPT}(\boldsymbol{q}_1,\boldsymbol{q}_2,k) =$$

$$\mathcal{K}^{(2)}_{\rm 2PPT}(\boldsymbol{q}_1,\boldsymbol{q}_2,k,\tau) = \frac{1}{\left(1+3\frac{\mathcal{H}^2}{q_1^2}\right)\left(1+3\frac{\mathcal{H}^2}{q_2^2}\right)}\Bigg[\Big(\beta_{\rm 2PPT}(k,\tau)-\alpha_{\rm 2PPT}(k,\tau)\Big)$$
$$+\frac{\beta_{\rm 2PPT}(k,\tau)}{2}\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)+\alpha_{\rm 2PPT}(k,\tau)\Big(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2\Big)^2$$
$$+\gamma_{\rm 2PPT}(k,\tau)\left(\frac{q_1}{q_2}-\frac{q_2}{q_1}\right)^2\Bigg]\,, \tag{7.90}$$

where

$$\alpha_{\rm 2PPT} = \frac{2}{7}+\frac{59\,\mathcal{H}^2}{14\,k^2}-\frac{27\,\mathcal{H}^4}{14\,k^4}\,, \tag{7.91}$$

$$\beta_{\rm 2PPT} = 1-\frac{\mathcal{H}^2}{2\,k^2}-\frac{18\,\mathcal{H}^4}{k^4}\,, \tag{7.92}$$

$$\gamma_{\rm 2PPT} = -\frac{3\,\mathcal{H}^2}{2\,k^2}-\frac{9\,\mathcal{H}^4}{2\,k^4}\,. \tag{7.93}$$

In deriving these expressions, which are now in a form similar to the ones used by Tram et. al. in Ref. [98] and in Chapter 4, we have made use of the following identities:

$$\frac{1}{q_1q_2} = \frac{1}{k^2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)+\frac{2}{k^2}\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2\,, \tag{7.94}$$

$$\frac{1}{q_1^2q_2^2} = \frac{1}{k^4}\left(4+\left(\frac{q_1}{q_2}-\frac{q_2}{q_1}\right)^2+4\,\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)+4(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\right)\,.$$

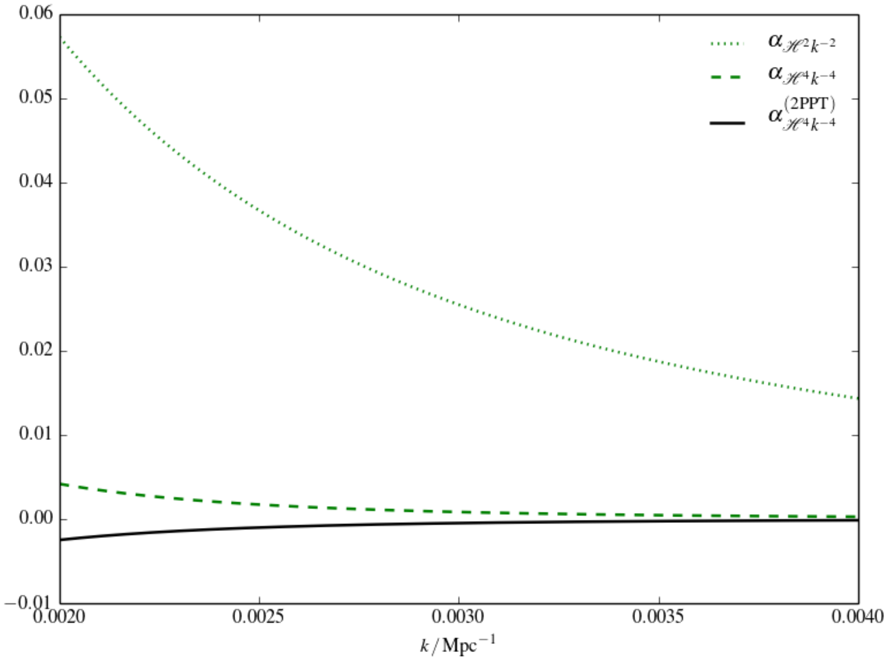


Figure 7.3.: Comparison of terms in $\alpha$ and $\alpha_{\mathrm{2PPT}}$ which scale as $\mathcal{H}^2 k^{-2}$ and $\mathcal{H}^4 k^{-4}$ respectively at the scale of interest.

The functions, $\alpha_{\mathrm{2PPT}}$, $\beta_{\mathrm{2PPT}}$ and $\gamma_{\mathrm{2PPT}}$ encode relativistic corrections in powers of $\dfrac{\mathcal{H}^2}{k^2}$. Comparison of the 2PPT coefficient functions with those from standard second-order CPT and reveals that differences arise at $\mathcal{O}\Big(\dfrac{\mathcal{H}^4}{k^4}\Big)$, i.e at extremely large scales.

The scale dependent terms in these functions are plotted in Figure 7.3, Figure 7.4 and Figure 7.5 and are compared to the equivalent terms in $\alpha$, $\beta$ and $\gamma$ in second order relativistic perturbation theory, as defined in [98].

It is clear from Figures 7.3 and 7.5 that terms scaling as $\frac{\mathcal{H}^2}{k^2}$ (plotted as green dots) remain an order of magnitude larger than those scaling as $\frac{\mathcal{H}^4}{k^4}$ (plotted as the solid green line for $\alpha$ and as the solid black line for $\alpha_{\mathrm{2PPT}}$ in both $\alpha$ and $\alpha_{\mathrm{2PPT}}$, at least down to scales of $k \sim 0.003\ \mathrm{Mpc}^{-1}$). We can therefore confidently expect the difference in $\alpha$ and $\alpha_{\mathrm{2PPT}}$ to be extremely small, at least down to these scales, and consequently that the approximation of this function is very good.

In the case of the Figure 7.4, we see that the magnitude of the terms in $\beta$ that scale as $\frac{\mathcal{H}^4}{k^4}$ (the solid green line) becomes larger than the magnitude of the terms that scale as $\frac{\mathcal{H}^2}{k^2}$ (the dashed green line) below scales of $k \sim 0.0026$. Accordingly, below these scales, $\beta_{\mathrm{2PPT}}$ will begin to significantly mis-estimate $\beta$.

It is worth noting though, that the terms scaling as $\frac{\mathcal{H}^2}{k^2}$ in the functions $\beta$ and $\beta_{\mathrm{2PPT}}$

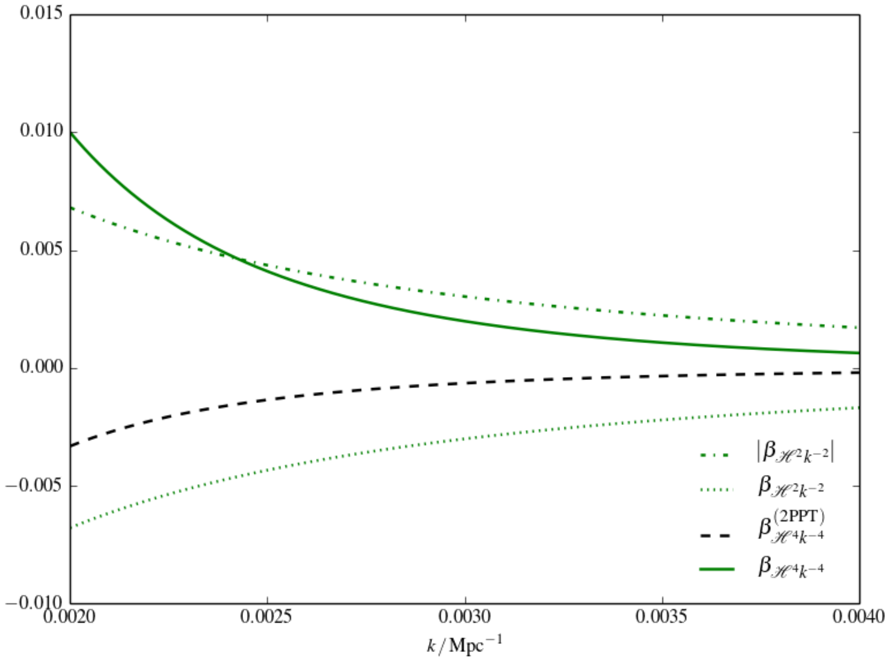


Figure 7.4.: Comparison of terms in $\beta$ and $\beta_{2\mathrm{PPT}}$ which scale as $\mathcal{H}^2k^{-2}$ and $\mathcal{H}^4k^{-4}$ respectively at the second scale of interest.

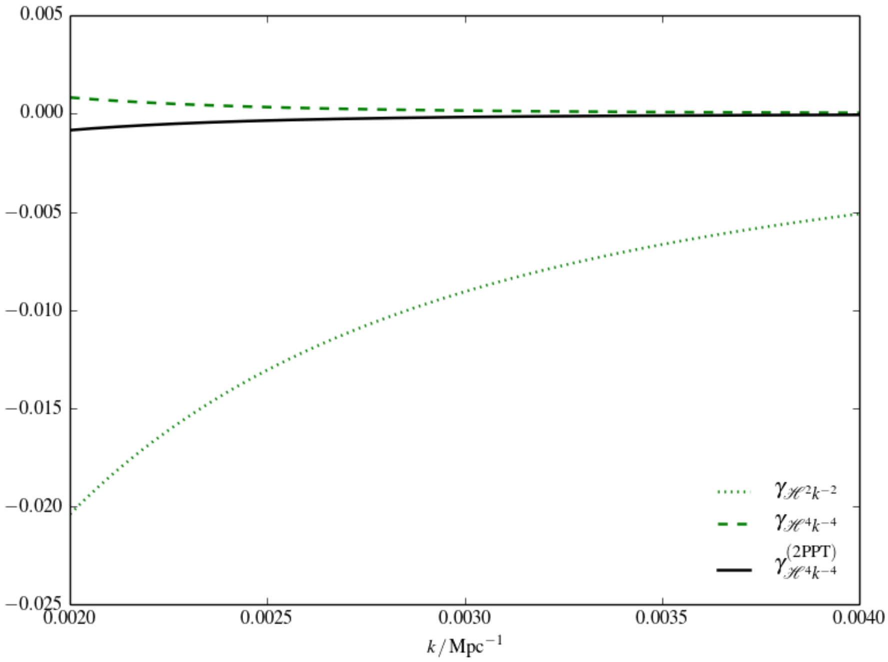


Figure 7.5.: Comparison of terms in $\gamma$ and $\gamma_{2\mathrm{PPT}}$ which scale as $\mathcal{H}^2k^{-2}$ and $\mathcal{H}^4k^{-4}$ respectively at the scale of interest.

at the scale $k \sim 0.003\ \mathrm{Mpc}^{-1}$ are still an order of magnitude smaller than the terms scaling as $\frac{\mathcal{H}^2}{k^2}$ in the functions $\alpha$, $\alpha_{\mathrm{2PPT}}$ and $\gamma$, $\gamma_{\mathrm{2PPT}}$, hence the resulting error in the density kernel $\mathcal{K}^{(2)}_{\mathrm{2PPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ should still be extremely small, provided that none of the magnitudes of the arguments $k_1$, $k_2$, $k_3$ are smaller than $k \sim 0.003\ \mathrm{Mpc}^{-1}$.

The bispectra for equilateral, squeezed and flattened configurations are shown in Figs. 7.6-7.8, respectively, along with the results from second-order CPT and NPT.

We can define the scale-dependent relative difference between the CPT bispectrum and the bispectra found using 2PPT,

$$E_{\mathrm{2PPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \left| \frac{B_{\mathrm{CPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) - B_{\mathrm{2PPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)}{B_{\mathrm{N}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)} \right| , \tag{7.95}$$

and also the relative difference between the CPT and NPT bispectra,

$$E_{\mathrm{N}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \left| \frac{B_{\mathrm{CPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) - B_{\mathrm{N}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)}{B_{\mathrm{N}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)} \right| . \tag{7.96}$$

These functions tell us how close the 2PPT and predictions are to the standard results from CPT - the smaller the value of $E_{\mathrm{2PPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$, the closer the result is to second-order CPT. We have chosen to normalise the difference by the value of the Newtonian bispectrum for a specific value of $k$. These differences are shown in Figs. 7.9, 7.11 and 7.13 for the equilateral, squeezed and flattened configurations.

Figures 7.9, 7.11 and 7.13 span many orders of magnitude. It is useful to focus on some particular scales of interest, in particular, to highlight where we expect sub-percent accuracy can be achieved. For this we compute the percentage difference between the 2PPT bispectrum and the CPT bispectrum in various $k$ regimes and configurations, defined as

$$\% = \left| \frac{B_{\mathrm{CPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) - B_{\mathrm{2PPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)}{B_{\mathrm{CPT}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)} \right| \cdot 100 . \tag{7.97}$$

Figure 7.10 demonstrates that the theoretical error remains at sub-percent above scales of $k \sim 0.002\ \mathrm{Mpc}^{-1}$ in the equilateral configuration. This is to be contrasted with the Newtonian approximation, for which the theoretical error is orders of magnitude higher at these scales.

The bottom panel of Figure 7.12 demonstrates that the theoretical error remains at sub-percent above scales of $k \sim 0.01\ \mathrm{Mpc}^{-1}$ in the squeezed configuration. This is to be contrasted with the Newtonian approximation, for which the theoretical error

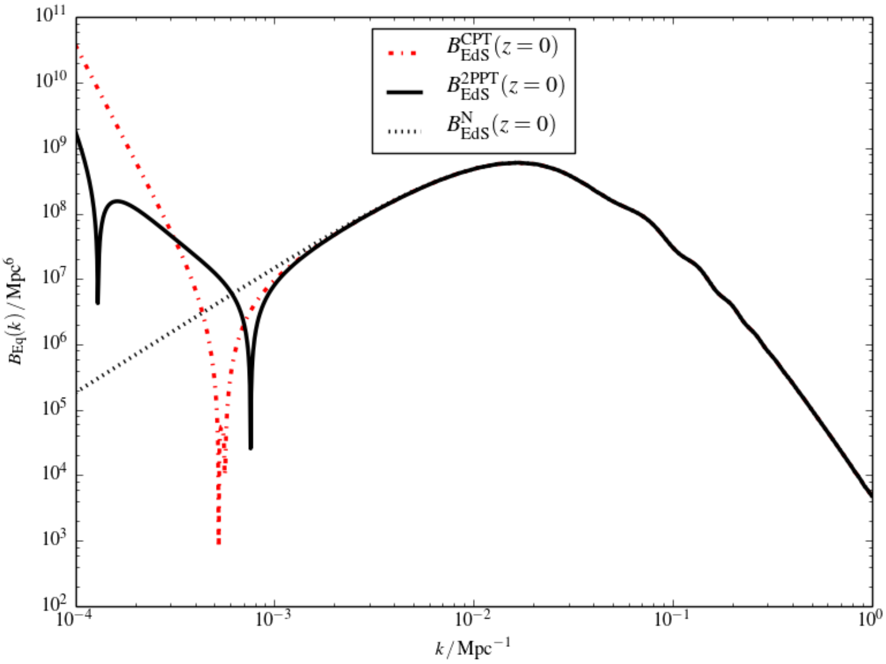


Figure 7.6.: The absolute value of the tree-level bispectrum induced by gravity for the equilateral configuration $B(k, k, k)$, in 2PPT, CPT and NPT.

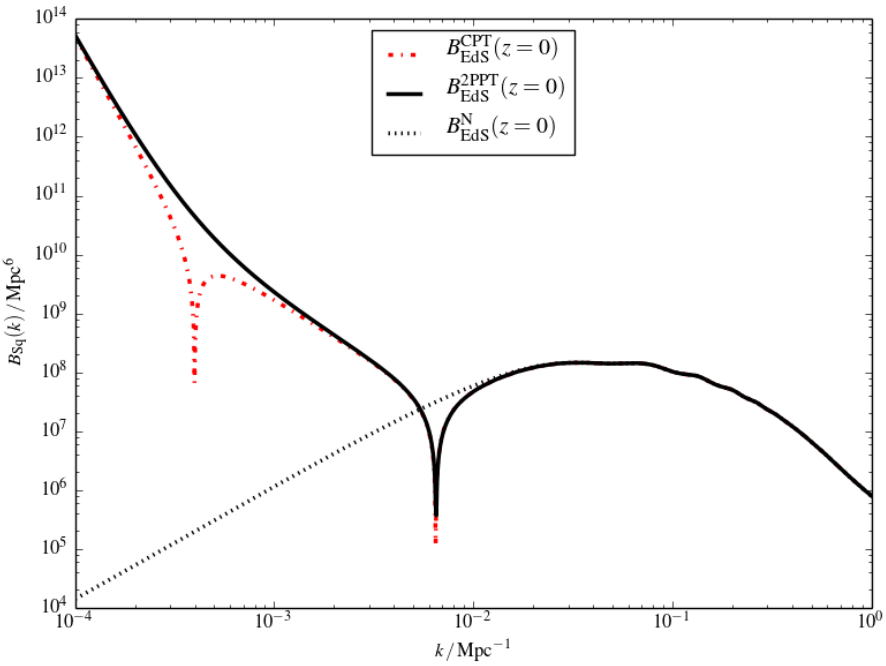


Figure 7.7.: The absolute value of the tree-level bispectrum induced by gravity for the squeezed configuration $B(k, k, k/16)$, in 2PPT, CPT and NPT.

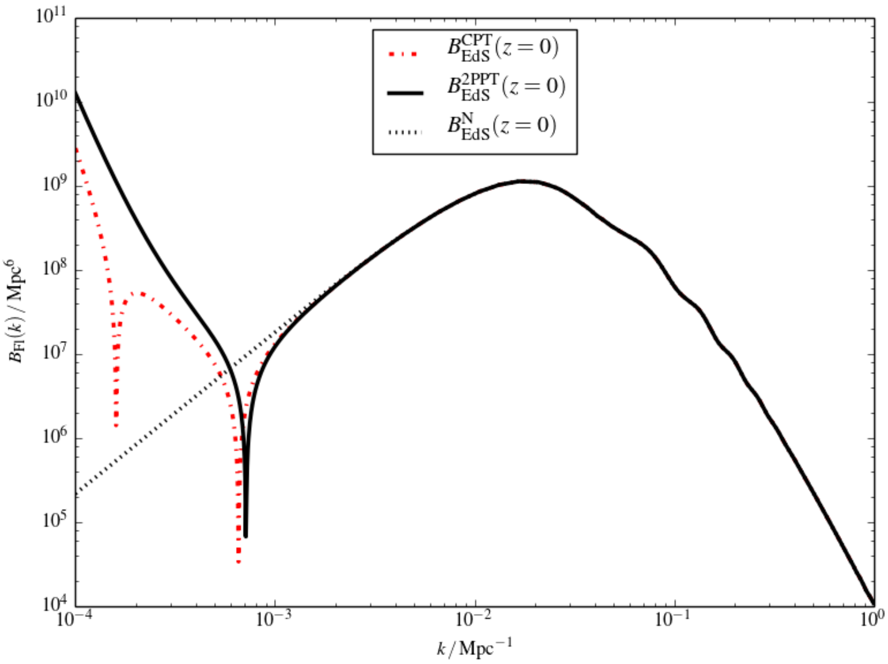


Figure 7.8.: The absolute value of the tree-level bispectrum induced by gravity for the flattened configuration $B(k, k, 2k)$, in 2PPT, CPT and NPT.

is orders of magnitude higher at these scales. The presence of a zero-crossing means that analysing the theoretical error is difficult on larger scales due to the divergence in the denominator.

We also focus on the percentage error compared to cosmological perturbation theory for the squeezed configuration in the regime $0.003\ \mathrm{Mpc}^{-1} < k < 0.006\ \mathrm{Mpc}^{-1}$ in the top panel of Figure 7.12. Whilst the rise in the theoretical error on smaller scales is due to the divergence, the rise in the error on large scales is fundamentally due to the fact that one of the arguments of the bispectrum, $k/16 \sim 0.003/16\ \mathrm{Mpc}^{-1} \sim 0.0002\ \mathrm{Mpc}^{-1}$, is being evaluated far outside the regime of applicability of the 2PPT approximation we considered by comparing the sizes of differently scaling terms in the coefficient functions $\alpha_{\mathrm{2PPT}}$, $\beta_{\mathrm{2PPT}}$, $\gamma_{\mathrm{2PPT}}$. We therefore conclude that 2PPT will only be useful constructing an order of magnitude estimate of the cosmological perturbation theory bispectrum in the range $0.001\ \mathrm{Mpc}^{-1} < k < 0.01\ \mathrm{Mpc}^{-1}$. The Newtonian approximation however, completely fails to predict the correct order of magnitude for the bispectrum in this range of scales, and so 2PPT significantly improves on the results of pure Newtonian perturbation theory.

Figure 7.14 demonstrates that the theoretical error remains at sub-percent above scales of $k \sim 0.0032\, h\,\mathrm{Mpc}^{-3}$ in the flattened configuration. This is to be contrasted with the Newtonian approximation, again for which the theoretical error is orders of

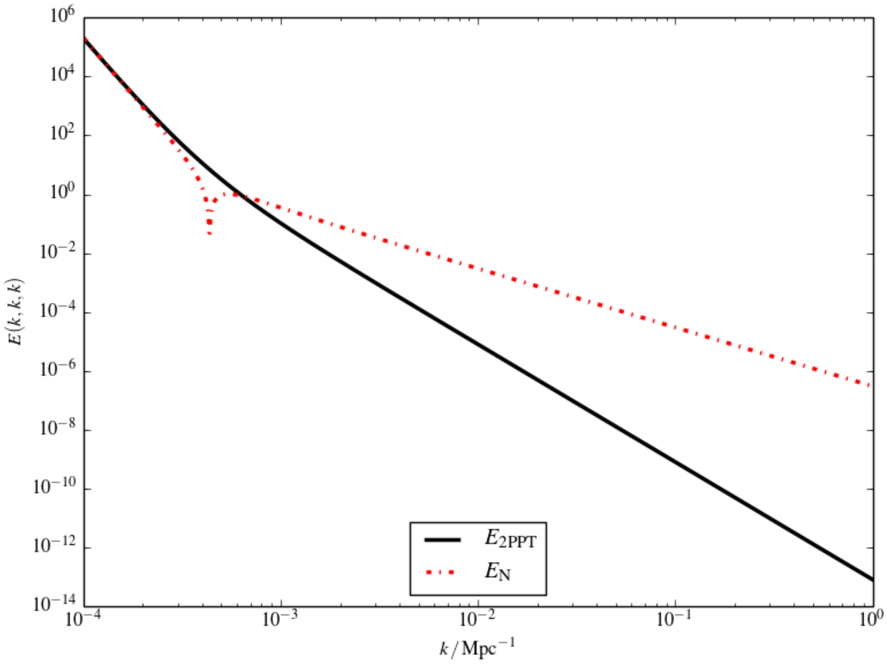


Figure 7.9.: The difference statistics $E_{2PPT}(k,k,k)$ and $E_N(k,k,k)$, for equilateral configuration.

magnitude higher at these scales. The 2PPT approximation performs slightly less well than in the case of the equilateral configuration, again due to the presence of smaller values in the argument of the bispectrum, resulting in theoretical error in the values of coefficient function $\beta_{2PPT}$, compared to $\beta$.

### 7.4.6. Discussion

Figures 7.6-7.8 demonstrate that there are significant differences between full second-order CPT and the second approximation to 2PPT. We emphasise that this is to be expected, as the field equations in each approach are different. The philosophy of the 2PPT approach is fundamentally different to cosmological perturbation theory, and this difference is illustrated in Figure 7.1. Rather than directly trying to approximate the full Einstein equation via linearisation, as is done in regular perturbation theory, the two-parameter expansion is used to derive a different set of equations for describing physics on multiple different length scales. The full 2PPT equations; however, contain nonlinear and inhomogeneous elements that make analytic progress difficult, so we have investigated using linearisation of the 2PPT equations to try and learn something about the physics they contain.

We believe it is important to stress that the relative success of the 2PPT scheme should *not* be judged on its proximity to second-order cosmological perturbation

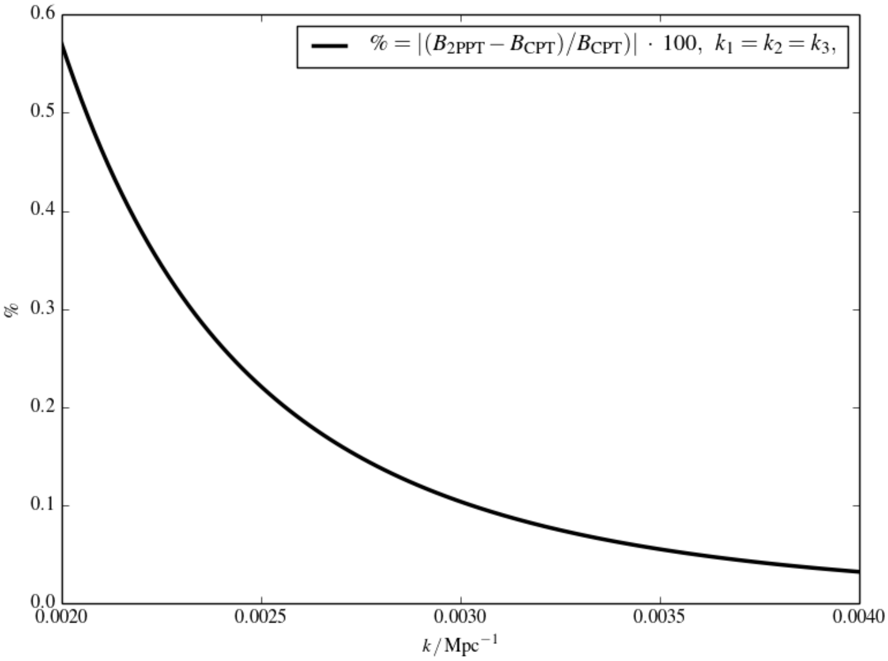


Figure 7.10.: The percentage error compared to cosmological perturbation theory for the equilateral configuration in the regime of interest.

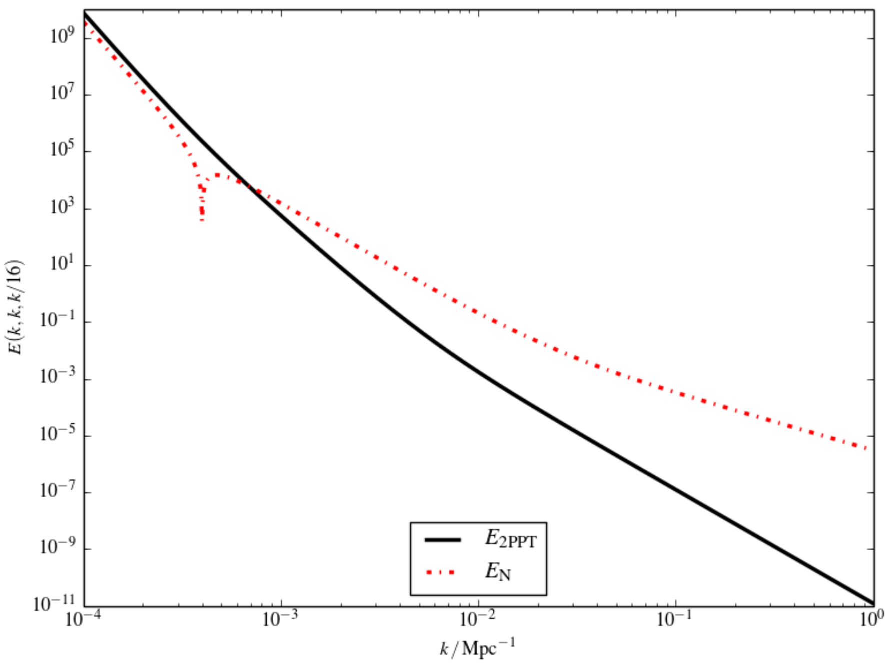


Figure 7.11.: The difference statistics $E_{2PPT}(k, k, k/16)$ and $E_N(k, k, k/16)$, for the squeezed configuration.

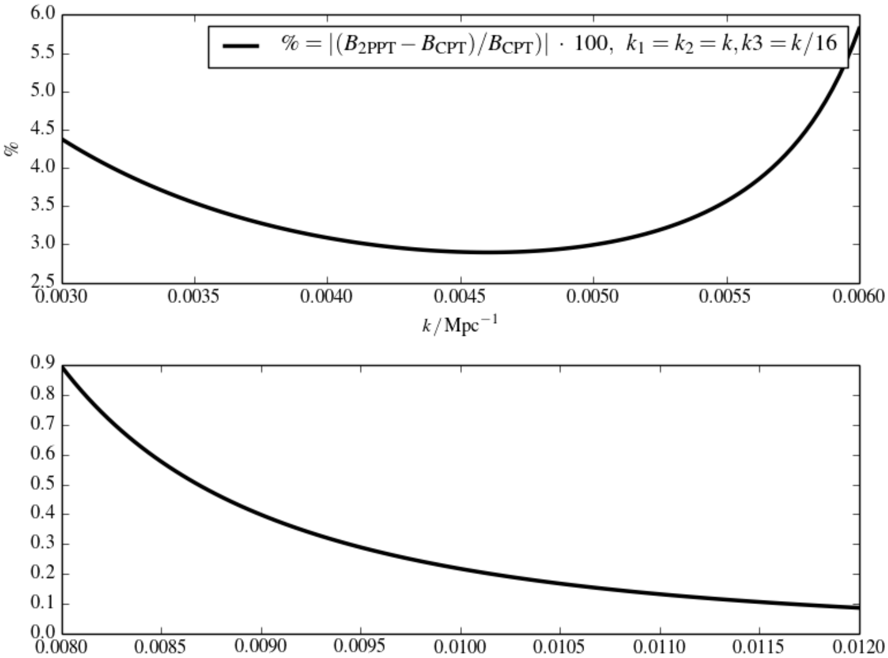


Figure 7.12.: The percentage error compared to cosmological perturbation theory for a moderately squeezed configuration in the two regime of interest, corresponding to scales of $0.003\ \mathrm{Mpc}^{-1} < k < 0.006\ \mathrm{Mpc}^{-1}$ and $k \sim 0.01\ \mathrm{Mpc}^{-1}$, for the top and bottom panels, respectively.

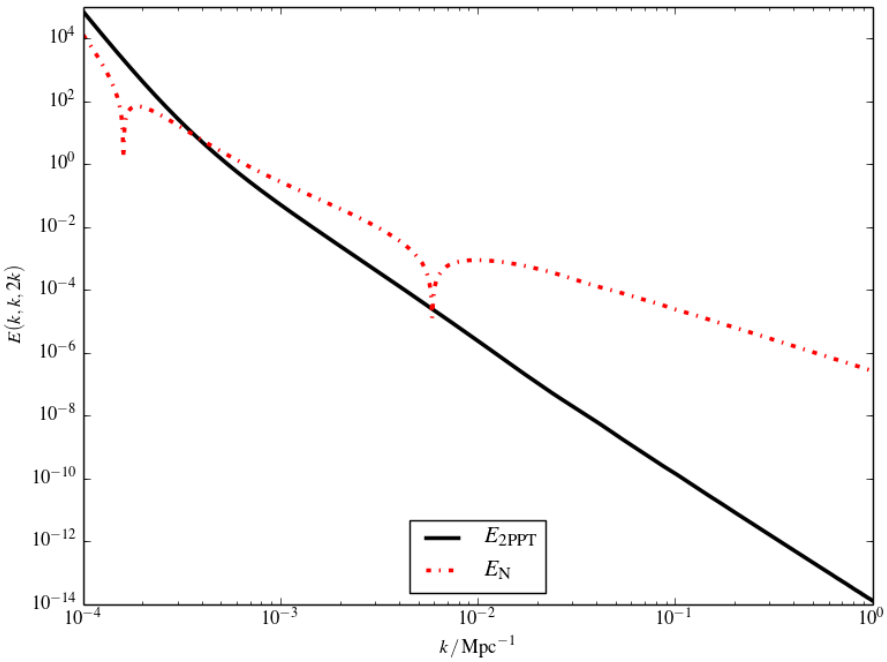


Figure 7.13.: The difference statistics $E_{\mathrm{2PPT}}(k, k, 2k)$ and $E_{\mathrm{N}}(k, k, 2k)$, for the flattened configuration.

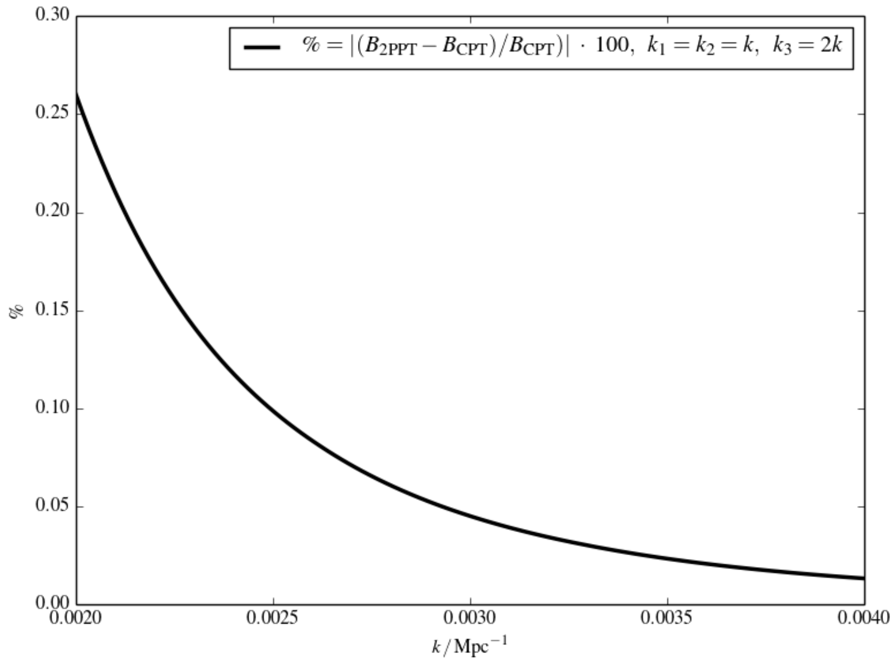


Figure 7.14.: The percentage error compared to cosmological perturbation theory for a flattened configuration in the regime of interest.

theory, since they are approximations to fundamentally different equations. Rather, this paper is an attempt to introduce the reader to a methodology for approximating solutions to the 2PPT equations that is similar to the methodology used in standard cosmological perturbation theory. The reader can be assured that we will always find that full second-order CPT results for large scales can be recovered simply by considering the $\mathcal{O}(\epsilon^2)$ 2PPT quantities, solving for them in an analogous fashion as we have done for the $\mathcal{O}(\epsilon)$ quantities here, and adding the results together. Rather, the second approximation to the 2PPT quantities that we have calculated here should themselves be understood as approximations to the *first-order* equations of cosmological perturbation theory, with corrections due to the existence of nonlinear structure on small scales.

It can be seen in Figures 7.9-7.13 that differences between the 2PPT bispectrum and the second-order CPT bispectrum are quite small for any scales that are not ultra-large (i.e $k > 10^{-3}\,\mathrm{Mpc}^{-1}$), with the possible exception of the flattened case. This is to be expected - the second approximation to 2PPT captures most, but not all of, the terms present in the full second-order field equations, because spatial derivatives of Newtonian terms appear at lower orders than they normally would, but time derivatives do not. We therefore expect that 2PPT will capture most of the interesting relativistic dynamics occurring on intermediate length scales

($0.003\,\mathrm{Mpc}^{-1} < k < 10^{-2}\,\mathrm{Mpc}^{-1}$). In the flattened case, one can see that differences between the 2PPT result and the second-order CPT result can become quite large around the $k \sim 10^{-3}\,\mathrm{Mpc}^{-1}$ scale. This is due to the zero-crossing occurring at slightly different values of $k$ in each case. Both the squeezed and equilateral configurations yield remarkably similar results in 2PPT to second-order CPT down to $k \gtrsim 0.003\,\mathrm{Mpc}^{-1}$. In particular, Figures 7.9-7.13 show that 2PPT is at least an order of magnitude closer to the full relativistic second-order CPT result than NPT alone in the region $0.003\,\mathrm{Mpc}^{-1} < k < 10^{-2}\,\mathrm{Mpc}^{-1}$.

In order to recover the terms most relevant at ultra-large scales, one would need to consider the 2PPT field equations for quantities of order $\sim \epsilon^2$ (i.e. at the level of second-order cosmological perturbations), and solve these equations in a fashion similar to the manner described in the previous sections. If we have to go through this convoluted setup procedure to calculate results that are relatively easily calculated in regular second-order CPT, without all of the information on the ultra-large scales, one might be tempted to ask why 2PPT is necessary at all. The answer is that 2PPT may yield significant advantages when trying to approximate quantities that require *third-order* calculation, at least on intermediate scales. In particular, calculating $P(k)_{1-\mathrm{loop}} = P(k)_{22}+P(k)_{13}$ in full relativistic CPT is an extremely challenging proposal, requiring a third-order calculation of $\delta_3$ in Poisson gauge. On the other hand, 2PPT naturally provides a framework in which only the most relevant terms from third-order are included. This may enable an estimation of relativistic effects in $P(k)_{1-\mathrm{loop}}$ down to $k \sim 0.003\,\mathrm{Mpc}^{-1}$, the scales likely to be accessed by next-generation surveys.

Let us examine this claim about the relative ease of calculation of $\delta^{(3)}_{2\mathrm{PPT}}$ (compared to $\delta^{(3)}$ in full relativistic perturbation theory) in a little more detail. In order to calculate a third approximation to the full solutions to Equations (7.12), (7.13), (7.14), (7.15), the first quantity we would have to consider would be

$$\frac{1}{3}\nabla^4\big(\psi^{(3)} - \phi^{(3)}\big) = \mathcal{S}^{(3)}_4 + \mathcal{S}^{(3)}_5 + \mathcal{S}^{(3)}_6\,. \tag{7.98}$$

Equipped with the knowledge that all first approximations to relativistic corrections (apart from $\delta^{(1)} = -2\varphi$ )are zero, we can simplify the forms of the quadratic source terms $\mathcal{S}^{(3)}_4$, $\mathcal{S}^{(3)}_5$ and $\mathcal{S}^{(3)}_6$ to containing only products of $\{U^{(2)}, \delta^{(2)}_N, \theta^{(2)}_N, \psi^{(2)}, \phi^{(2)}, \theta^{(2)}, \delta^{(2)}\}$ with $\{U^{(1)}, \delta^{(1)}_N, \theta^{(1)}_N, \delta^{(1)}\}$, and any cubic products of first order terms that are introduced. The only cubic first order terms appearing in the field equations will come from the terms $(1+\delta^{(1)}_N)v^{(1)^2}_N$.

This situation is simpler than the one arising in full third order relativistic perturbation theory, where a third order calculation of all field equations and source terms is required. There are a significantly larger number of these terms, rendering the calculation more cumbersome and difficult.

There are other difficulties associated with the calculation of relativistic effects in the 1-loop power spectrum aside from simply the number of terms involved. In Newtonian perturbation theory, Galilean invariance is known to guarantee the cancellation of separate IR divergences that appear in the convolution integrals contributing to $P_{22}$ and $P_{13}$. This cancellation is not guaranteed to happen in relativistic perturbation theory, indeed as is found by the authors in [137] when considering this calculation in the context of the gradient expansion approach. Without performing the calculation directly we cannot comment on the specifics of IR divergences, however we do anticipate being able to implement (at least in a worst-case scenario) a hard cut-off at very small values of $k$ as is done in [137].

One must remember that we have calculated only an approximation to 2PPT, where linearisation of the Newtonian equations is feasible. In the real late universe, significant nonlinearity should be present on the shortest scales, and one would expect precisely the terms highlighted by 2PPT to be dominant when looking at their effect on intermediate scales. In such a universe, the (all orders) 2PPT equations should be expected to provide a better prediction than pure CPT equations (if they could be solved at all), as the 2PPT approach was designed with exactly this problem in mind.

In the next chapter, we will extend the approximation scheme presented here to the case of a $\Lambda$CDM universe.

# 8. Approximate solutions to 2PPT in $\Lambda$CDM universes

In this chapter, we address the problem of extending the method of approximating solutions introduced in Chapter 7 to universes experiencing accelerated expansion driven by a cosmological constant. The methodology used is largely the same as that employed in Chapter 7, although a few additional considerations are necessary. All the work presented in this chapter is of my own doing, and will subsequently be submitted for publication in an upcoming paper.

## 8.1. Motivation

In order to model the late universe realistically, it is necessary to account for late-time accelerated expansion. The standard method for including this effect is to include a cosmological constant in the Einstein field equations, as described in Chapter 2. Current data from cosmic microwave background anisotropy (see [74]) strongly favours the $\Lambda$CDM paradigm, and so understanding the growth of structure in $\Lambda$CDM geometry is extremely important for upcoming galaxy surveys. As such, extending the approximation scheme put forward in Chapter 7 to the case of a $\Lambda$CDM geometry is extremely desirable.

This extension is non-trivial, since many results that do not involve time-dependencies in the Einstein-de Sitter case (or at the very least have simple time dependencies e.g. $\propto a$) will subsequently become time-dependent in the $\Lambda$CDM case. This complicates the algebraic manipulations in places, and results in an ordinary differential equation that must be numerically integrated in order to find a particular solution to the second approximation to the gravitational potential.

## 8.2. Set-up

The metric and the stress energy tensor remain unchanged from Chapter 7; we repeat them here for convenience:

$$ds^2 = a(\tau)\Big[-(1-2U-2\phi)d\tau^2 + (1-2U-2\psi)\delta_{ij}dx^i dx^j\Big] \,, \tag{8.1}$$

$$T^{\mu\nu} = (\rho_{\rm N} + \delta\rho)u^\mu u^\nu \,. \tag{8.2}$$

The only difference from Chapter 7 is the inclusion of the cosmological constant in the field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \,. \tag{8.3}$$

This results in slightly modified field equations. At leading order we have

$$\mathcal{H}' = -\frac{4\pi a^2}{3}\rho_{\rm N} - \frac{1}{3}\nabla^2 U + \frac{1}{3}\Lambda a^2 \,, \tag{8.4}$$

$$\mathcal{H}^2 = \frac{8\pi a^2}{3}\rho_{\rm N} + \frac{2}{3}\nabla^2 U + \frac{1}{3}\Lambda a^2 \,, \tag{8.5}$$

which can be separated into homogeneous and inhomogeneous components via the usual averaging procedure, leaving

$$\mathcal{H}' = -\frac{4\pi a^2\bar{\rho}}{3} + \frac{1}{3}\Lambda a^2 \,, \tag{8.6}$$

$$\mathcal{H}^2 = \frac{8\pi a^2\bar{\rho}}{3} + \frac{1}{3}\Lambda a^2 \,, \tag{8.7}$$

$$\nabla^2 U = 4\pi a^2\bar{\rho}\delta_{\rm N} \,. \tag{8.8}$$

The Friedmann and acceleration equations can be solved in the normal way, yielding the familiar solution for the scale factor given in Chapter 2. The inhomogeneous component of the leading order system is completed by the normal Newtonian non-linear stress-energy conservation equations,

$$\delta_{\rm N}' + \theta_{\rm N} = -\partial^i\big(\delta_{\rm N} v_{{\rm N}i}\big) \,, \tag{8.9}$$

$$\theta_{\rm N}' + \mathcal{H}\theta_{\rm N} + \frac{3\mathcal{H}^2}{2}\delta_{\rm N} = -\partial^i\big(v_{{\rm N}j}\partial^j v_{{\rm N}i}\big) \,, \tag{8.10}$$

where $\theta_{\rm N} = \partial^i v_{{\rm N}i}$ is the normal velocity divergence. The subleading order evolution equation is now modified to become

$$\begin{aligned}
&(\psi + U)'' + 3\mathcal{H}(\psi + U)' + a^2\Lambda(\psi + U) \\
&\quad = \frac{4\pi a^2\bar{\rho}}{3}(1 + \delta_{\rm N})v_{\rm N}^2 + \mathcal{H}(\psi' - \phi') + \frac{1}{3}\nabla^2(\psi - \phi) \\
&\quad + \frac{7}{6}(\nabla U)^2 + \frac{2}{3}(\phi + \psi + 2U)\nabla^2 U + a^2\Lambda(\psi - \phi) \;,
\end{aligned} \tag{8.11}$$

whilst the constraint equations remains the same as Equations (7.13), (7.14) and (7.15) in Chapter 7.

## 8.3. First approximation

We begin by inserting our familiar Newtonian perturbation theory series decompositions

$$\delta_{\rm N} = \delta_{\rm N}^{(1)} + \frac{1}{2}\delta_{\rm N}^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\delta_{\rm N}^{(n)}}{n!} \tag{8.12}$$

$$\theta_{\rm N} = \theta_{\rm N}^{(1)} + \frac{1}{2}\theta_{\rm N}^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\theta_{\rm N}^{(n)}}{n!} \tag{8.13}$$

$$U = U^{(1)} + \frac{1}{2}U^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{U^{(n)}}{n!} \;, \tag{8.14}$$

into Equations (8.9) and (8.10). We insert the analogous series decompositions for the cosmological variables,

$$\delta = \delta^{(1)} + \frac{1}{2}\delta^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\delta^{(n)}}{n!} \tag{8.15}$$

$$\theta = \theta^{(1)} + \frac{1}{2}\theta^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\theta^{(n)}}{n!} \tag{8.16}$$

$$\psi = \psi^{(1)} + \frac{1}{2}\psi^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\psi^{(n)}}{n!} \tag{8.17}$$

$$\phi = \phi^{(1)} + \frac{1}{2}\phi^{(2)} + \cdots = \sum_{n=1}^{\infty} \frac{\phi^{(n)}}{n!} \;, \tag{8.18}$$

into Equations (8.11) and (7.13), (7.14) and (7.15). Let us first consider the first approximation to the leading order Newtonian system. It is easy to verify that we recover the standard evolution equation for the linearised Newtonian density contrast:

$$\delta_{\rm N}^{(1)\prime\prime} + \mathcal{H}\delta_{\rm N}^{(1)\prime} - \frac{3\mathcal{H}_0^2\Omega_{m0}}{2a}\delta_{\rm N}^{(1)} = 0 \; , \tag{8.19}$$

which is solved by the standard growth factor $\delta_{\rm N}^{(1)} = \mathcal{D}(a)\delta_{\rm N}^{(1)}(\boldsymbol{x})$. This result implies that $U^{(1)}$ satisfies the usual differential equation, Equation (4.55),

$$U^{(1)\prime\prime} + 3\mathcal{H}U^{(1)\prime} + a^2\Lambda U^{(1)} = 0 \; , \tag{8.20}$$

which is solved by $U^{(1)} = \varphi = g(\tau)\varphi_0(\boldsymbol{x})$. Similarly, at first approximation, the cosmological 2PPT equations reduce to

$$\psi^{(1)} = \phi^{(1)} \; , \tag{8.21}$$

and

$$(\psi^{(1)} + U^{(1)})'' + 3\mathcal{H}(\psi^{(1)} + U^{(1)})' + a^2\Lambda(\psi^{(1)} + U^{(1)}) = 0 \; . \tag{8.22}$$

This equation is also solved by the usual time dependency, $g(\tau)$, so we choose to implement the same re-definition employed in Chapter 7 in Equations (7.47), leading us to extend the domain of $\varphi(\boldsymbol{x})$ to the long wavelength regime, and regard $\{\psi^{(1)}, \phi^{(1)}, \delta^{(1)}, \theta^{(1)}\}$ as the relativistic extensions to the Newtonian solutions. Accordingly we choose

$$\psi^{(1)} = 0 \qquad \Longrightarrow \delta^{(1)} = -2\varphi \; , \tag{8.23}$$

and

$$\theta^{(1)} = 0 \; . \tag{8.24}$$

The motivation for these choices remains the same as in Chapter 7. Given these results, it is easy to see that the first approximation to 2PPT gives identical results to standard first-order perturbation theory in ΛCDM.

## 8.4. Second approximation

At second approximation, the evolution equation becomes

$$
\begin{aligned}
&(\psi^{(2)}+U^{(2)})''+3\mathcal{H}(\psi^{(2)}+U^{(2)})'+a^2\Lambda(\psi^{(2)}+U^{(2)})\\
&=\frac{8\pi a^2\bar{\rho}}{3}(v_{\mathrm{N}}^{(1)})^2+\mathcal{H}(\psi^{(2)}-\phi^{(2)})'+\frac{1}{3}\nabla^2(\psi^{(2)}-\phi^{(2)})\\
&\quad+\frac{7}{3}(\nabla\varphi)^2+\frac{8}{3}\varphi\nabla^2\varphi+a^2\Lambda(\psi^{(2)}-\phi^{(2)})\ .
\end{aligned}
\tag{8.25}
$$

The similarities between Equation (8.25) and Equation (4.74) from Chapter 4 should not be understated. However, whilst in Chapter 7 the evolution equation for the gravitational scalar potential was found to be identical, it is already apparent that there are small differences between Equation (8.25) and Equation (4.74). In particular Equation (4.74) includes the terms $8\mathcal{H}\varphi\varphi'$, $(\varphi')^2$, and $4a^2\Lambda\varphi^2$, all of which are absent in Equation 8.25 due to the use of the two-parameter counting scheme in the derivation of the original equation. The implication of this is that the evolution of the gravitational potentials will be different in 2PPT and CPT in the ΛCDM universe, compared to the Einstein-de Sitter universe where they were the same as cosmological perturbation theory in 2PPT.

### 8.4.1. Constraint

We proceed, as usual, by isolating the combination $\psi^{(2)}-\phi^{(2)}$ using the trace-free $ij$ field equation. Fortunately, the inclusion of the cosmological constant in the field equation does not affect this equation, so the results derived in Chapter 7 still hold. We can therefore immediately write down

$$
\begin{aligned}
\nabla^2\nabla^2\psi^{(2)}=&\ \nabla^2\nabla^2\phi^{(2)}-4g^2\nabla^2\nabla^2\varphi_0\\
&-8g^2\left(\frac{f^2}{\Omega_m}+\frac{3}{2}\right)\nabla^2\nabla^2\boldsymbol{\Theta}_0\\
&-4g^2\varphi_0^2+Q\ ,
\end{aligned}
\tag{8.26}
$$

by analogy with the standard second-order perturbation theory of Chapter 4. Here, $Q$, $P^i{}_j$, $P$ and $N$ all remain unchanged from their definitions in Chapter 4. Now using the standard results

$$\begin{aligned}
-4a^2\Lambda &= 4\mathcal{H}^2\Omega_m\left(1-\frac{1}{\Omega_m}\right) ,\\
\mathcal{H}Q' + a^2\Lambda Q &= 12g^2\mathcal{H}^2\Omega_m\left(2\frac{(f-1)^2}{\Omega_m} - \frac{3}{\Omega_m} + 3\right)\boldsymbol{\Theta}_0 ,\\
N &= \frac{4}{3}g^2\left(\frac{f^2}{\Omega_m} + \frac{3}{2}\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right) ,\\
-8\mathcal{H}\varphi\varphi' &= -8\mathcal{H}^2g^2(f-1)\varphi_0^2 ,
\end{aligned} \tag{8.27}$$

we can rewrite Equation (8.25) in the form

$$\begin{aligned}
&(\psi^{(2)} + U^{(2)})'' + 3\mathcal{H}(\psi^{(2)} + U^{(2)})' + a^2\Lambda(\psi^{(2)} + U^{(2)})\\
= \;& -4\mathcal{H}^2g^2\left(2(f-1) + \Omega_m\left(1-\frac{1}{\Omega_m}\right)\right)\varphi_0^2\\
&+ 12g^2\mathcal{H}^2\Omega_m\left(2\frac{(f-1)^2}{\Omega_m} - \frac{3}{\Omega_m} + 3\right)\boldsymbol{\Theta}_0\\
&+ \frac{4}{3}g^2\left(\frac{f^2}{\Omega_m} + \frac{3}{2}\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right) - g^2(\nabla\varphi_0)^2 .
\end{aligned} \tag{8.28}$$

## 8.4.2. Evolution

Now, it can be shown without difficulty that

$$\begin{aligned}
&U^{(2)\prime\prime} + 3\mathcal{H}U^{(2)\prime} + a^2\Lambda U^{(2)}\\
&= \frac{4}{3}g^2\left(\frac{f^2}{\Omega_m} + \frac{3}{2}\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right) - g^2(\nabla\varphi_0)^2 ,
\end{aligned} \tag{8.29}$$

is solved by the standard Newtonian solution

$$U^{(2)} = \frac{2\mathcal{D}^2}{3a\mathcal{H}_0\Omega_{m0}}\partial_i\varphi_0\partial^i\varphi_0 - \frac{4(\mathcal{D}^2+\mathcal{F})}{3a\mathcal{H}_0\Omega_{m0}}\boldsymbol{\Psi}_0 , \tag{8.30}$$

where the definitions of $\mathcal{F}$ and $\boldsymbol{\Psi}_0$ remain the same as in Chapter 4. This result demonstrates that the Newtonian evolution of the system is unaffected by the two-parameter approach. Subtracting the Newtonian result away leaves

$$
\begin{aligned}
&\psi^{(2)\prime\prime} + 3\mathcal{H}\psi^{(2)\prime} + a^2\Lambda\psi^{(2)} \\
&= -4\mathcal{H}^2 g^2\left(2(f-1) + \Omega_m\left(1 - \frac{1}{\Omega_m}\right)\right)\varphi_0^2 \\
&+ 12 g^2\mathcal{H}^2\Omega_m\left(2\frac{(f-1)^2}{\Omega_m} - \frac{3}{\Omega_m} + 3\right)\boldsymbol{\Theta}_0 \, .
\end{aligned}
\tag{8.31}
$$

By analogy to Chapter 4, we can immediately write down the solution

$$
\psi^{(2)} = \mathcal{B}_1\varphi_0^2 + 6\mathcal{B}_2\boldsymbol{\Theta}_0 + \frac{g}{g_{in}}\psi^{(2)}_{in} \, , \tag{8.32}
$$

where $b_n = a\mathcal{B}_n$, and the $b_n$ satisfy

$$
b_1^{\mathrm{p}\prime\prime} + \mathcal{H}b_1^{\mathrm{p}\prime} - \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}b_1^{\mathrm{p}} = -\frac{4\mathcal{H}^2\mathcal{D}^2}{a}\left(2(f-1) + (\Omega_m - 1)\right) \, , \tag{8.33}
$$

$$
b_2^{\prime\prime} + \mathcal{H}b_2^{\prime} - \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}b_2 = \frac{2\mathcal{H}^2\mathcal{D}^2}{a}\left(2(f-1)^2 + 3(\Omega_m - 1)\right) \, . \tag{8.34}
$$

It is immediately apparent that the solution for $b_2$ is unchanged from the one given in Chapter 4,

$$
b_2 = -2\mathcal{D}(g_{in} - g) \, . \tag{8.35}
$$

The solution for $b_1^{\mathrm{p}}$ however is a different matter - no easy analytic solution can be found, however a formal solution can be constructed in terms of integrations of conformal time. The first step is to consider the two independent solutions of the homogeneous equation

$$
p^{\prime\prime} + \mathcal{H}p^{\prime} - \frac{3}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}p = 0 \, , \tag{8.36}
$$

which are given by $p_1(\tau) = \mathcal{D}$ and $p_2(\tau) = \frac{\mathcal{H}}{a}$. We can construct particular solutions by considering integrations of the source function together with combinations of the Wronskian of these independent solutions. If we define a source function

$$
\mathcal{I}(\tau) = \frac{\mathcal{H}^2\mathcal{D}^2}{a}\left(-8f - 4\Omega_m + 12\right) \, , \tag{8.37}
$$

and construct the Wronskian,

$$\mathcal{W}(\tau) = p_1 p_2' - p_1' p_2 \ , \tag{8.38}$$

$$= -\frac{\mathcal{H}^2 \mathcal{D}}{a}\left(f + \frac{3}{2}\Omega_m\right) \ , \tag{8.39}$$

then a particular solution to Equation (8.33) is given by

$$b_1^{\mathrm{P}} = -p_1(\tau)\int_{\tau_{in}}^{\tau} \frac{p_2(\tau)\,\mathcal{I}(\tau)}{\mathcal{W}(\tau)}\,\mathrm{d}\tau + p_2(\tau)\int_{\tau_{in}}^{\tau} \frac{p_1(\tau)\,\mathcal{I}(\tau)}{\mathcal{W}(\tau)}\,\mathrm{d}\tau \ , \tag{8.40}$$

via the method of variation of parameters. It is easy to see that this result reduces to the correct Einstein-de Sitter limit, namely $b_2^{\mathrm{EdS}} = 0$, since the source function $\mathcal{I}(\tau) \to 0$ in the same limit. Whilst this result is cumbersome to deal with analytically, it can be evaluated numerically without too much difficulty. By writing $\mathrm{d}a = a\mathcal{H}\mathrm{d}\tau$, we write the integral in terms of the scale factor

$$b_1^{\mathrm{P}} = -p_1(a)\int_{a_{in}}^{a} \frac{p_2(a)\,\mathcal{I}(a)}{a\mathcal{H}\mathcal{W}(a)}\,\mathrm{d}a + p_2(a)\int_{a_{in}}^{a} \frac{p_1(a)\,\mathcal{I}(a)}{a\mathcal{H}\mathcal{W}(a)}\,\mathrm{d}a \ . \tag{8.41}$$

How should we interpret $\tau_{in}$ and $a_{in}$ in these expressions? In Section 7.4.4 we argued that we should set the initial conditions for 2PPT using the standard initial conditions from second order perturbation theory, since the universe is expected to be well described by such an approximation, at least until the growth of significant nonlinear structure on small-scales. Since structure formation begins in the matter-dominated era (where $b_1^{\mathrm{EdS}} \to 0$), we should interpret $\tau_{in}$ as being the *crossover* time described in Section 7.4.4. Since the value of $b_1$ should remain extremely small until the effects of the cosmological constant become relevant, there should be no problem with taking

$$g(\tau_{in}) = g(\tau_{\mathrm{cross}}) \ . \tag{8.42}$$

The function $g(z)$ approaches a constant value in matter-domination, given by $g_{in}$. Accordingly, we are free to select any moment of time of our choosing as the crossover time, provided that time is well within the matter-dominated era. Since we are free to add the homogeneous solution to $b_1^{\mathrm{2PPT}}$ (multiplied by constants), without changing the fact that the sum of a particular solution and homogeneous solution is a different particular solution, we choose to ensure continuity of the metric at $\tau_{in}$

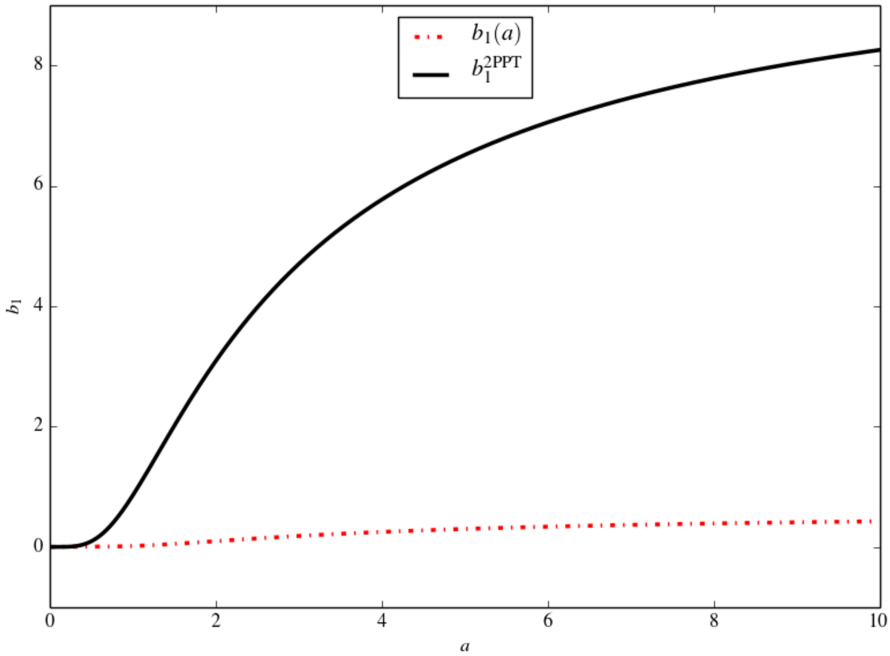


Figure 8.1.: A plot of the $b_1^{2\mathrm{PPT}}(a)$ solution, compared to the $b_1(a)$ solution from second order perturbation theory. The values of both functions are calculated via the numerical integration method given above.

by setting

$$b_1^{2\mathrm{PPT}}(a) = b_1^{\mathrm{p}}(a) - \frac{b^{\mathrm{p}}(a_{in}) - b_1(a_{in})}{\mathcal{D}(a_{in})}\mathcal{D}(a) \; . \tag{8.43}$$

This ensures that $b_1^{2\mathrm{PPT}}(a_{in}) = b_1(a_{in})$ at the crossover time (where $b_1(a)$ is the solution to the equivalent equation for $b_1$ in normal second-order cosmological perturbation theory) and therefore that the metric is continuous. We choose a redshift of $z = 19$ for $z_{\mathrm{cross}}$ which is equivalent to $a_{in} = 0.05$. It is easy to see that $b_1^{2\mathrm{PPT}} \to 0$ still holds, as in this limit both $b^{\mathrm{p}}$ and $b_1$ also go to zero.

It can be seen that $b_1^{2\mathrm{PPT}}(a)$ is significantly enhanced (by approximately an order of magnitude) compared to the standard case in cosmological perturbation theory at late times. This situation is to be contrasted with that which arose in Chapter 7, where the solutions for the evolution of the metric were identical in cosmological perturbation theory and the second approximation to two-parameter perturbation theory. This difference can be attributed to the absence of the time derivatives of the first order solutions in the two-parameter scheme. Comparing the source function

$\mathcal{I}$, with the equivalent construction in normal perturbation theory,

$$\mathcal{I}_{\rm CPT}(\tau) = \frac{\mathcal{H}^2\mathcal{D}^2}{a}\left(f^2 - 2f + 1\right) , \tag{8.44}$$

it is easy to see that when $f < 1$, $|\mathcal{I}| > |\mathcal{I}_{\rm CPT}$, which in turn forces the value of the integrals in the particular solution to be larger than they would be in cosmological perturbation theory.

## 8.5. Fourier kernels

### 8.5.1. Gravitational potential kernel

In Chapter 7 the solution for the evolution of the scalar gravitational potential was identical to that derived in normal second-order cosmological perturbation theory, with differences in the solution for the dark matter overdensity only arising as a result of a modified scalar constraint equation. In the ΛCDM case, this no longer remains true. As seen in Section 8.4.2, the solution to the evolution equation is modified, meaning that the second approximation to the 2PPT gravitational potential, $\psi^{(2)} + U^{(2)}$, will have different statistical correlation functions to the second order gravitational potential $\Psi_2$ in cosmological perturbation theory. As such, we will investigate the leading order bispectrum of the metric potential $\psi^{(2)} + U^{(2)}$.

In order to calculate the bispectrum of the gravitational potential, we must Fourier transform the solutions. We will write the second-order cosmological perturbation theory solution from Chapter 4 as

$$\Psi_2 = C_1\varphi_0^2 + C_2\boldsymbol{\Theta}_0 + C_3\boldsymbol{\Psi}_0 + C_4(\nabla\varphi_0)^2 , \tag{8.45}$$

where

$$C_1 = -g^2 + \frac{5}{3}gg_{in}(1 - 2a_{\rm NL}) + \frac{2\mathcal{H}^2 f^2}{3\mathcal{H}_0^2\Omega_{m0}}\mathcal{D}g , \tag{8.46}$$

$$C_2 = 6\left(2g^2 - \frac{10}{3}gg_{in}\right) , \tag{8.47}$$

$$C_3 = -\frac{4}{3a\mathcal{H}_0^2\Omega_{m0}}(\mathcal{D}^2 + \mathcal{F}) , \tag{8.48}$$

$$C_4 = \frac{2}{3a\mathcal{H}_0^2\Omega_{m0}}\mathcal{D}^2 . \tag{8.49}$$

Taking the Fourier transform of this expression, we obtain

$$\Psi_2(\boldsymbol{k}) = \int \mathrm{d}^3q_1\mathrm{d}^3q_2\, \delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\times \\ \Bigg[C_1 - \frac{C_2}{2k^4}q_1^2q_2^2\Big(1-(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\Big) - \frac{C_2}{3k^2}\,q_1q_2(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2) \\ + \frac{C_3}{2k^2}\,q_1^2q_2^2\Big(1-(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\Big) + C_4\,q_1q_2(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)\Bigg]\varphi_0(q_1)\varphi_0(q_2)\,. \tag{8.50}$$

This allows us to immediately write down an expression for the kernel for $\Psi_2$,

$$\mathcal{K}_2^{\Psi} = \Bigg[C_1 - \frac{C_2}{2k^4}q_1^2q_2^2\Big(1-(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\Big) - \frac{C_2}{3k^2}\,q_1q_2(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2) \\ + \frac{C_3}{2k^2}\,q_1^2q_2^2\Big(1-(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\Big) + C_4\,q_1q_2(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)\Bigg]\,. \tag{8.51}$$

The 2PPT version of this kernel is found by simply replacing $C_1$ with the equivalent expression from 2PPT, which is found to be

$$C_1^{\mathrm{2PPT}} = 2gg_{in}\Big(-\frac{5}{3}(a_{\mathrm{NL}}-1)-1\Big) + \frac{b_1^{\mathrm{2PPT}}}{a}\,, \tag{8.52}$$

leading to the modified two-parameter kernel

$$\mathcal{K}_2^{\psi+U} = \Bigg[C_1^{\mathrm{2PPT}} - \frac{C_2}{2k^4}q_1^2q_2^2\Big(1-(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\Big) - \frac{C_2}{3k^2}\,q_1q_2(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2) \\ + \frac{C_3}{2k^2}\,q_1^2q_2^2\Big(1-(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)^2\Big) + C_4\,q_1q_2(\hat{\boldsymbol{q}}_1\cdot\hat{\boldsymbol{q}}_2)\Bigg]\,. \tag{8.53}$$

### 8.5.2. Dark matter overdensity kernel

Given all constituent parts of our solution in Equation (8.32), we proceed as was done in Chapter 7, and use

$$\frac{1}{3}\nabla^2\psi^{(2)} - \mathcal{H}(\psi^{(2)}+U^{(2)})' - \mathcal{H}^2(\psi^{(2)}+U^{(2)}) \\ = \frac{1}{2}\frac{\mathcal{H}_0^2\Omega_{m0}}{a}\delta^{(2)} + \frac{\mathcal{H}_0^2\Omega_{m0}}{a}(v_N^{(1)})^2 - \frac{8}{3}g^2\varphi_0\nabla^2\varphi_0 - g^2\nabla(\varphi_0)^2\,, \tag{8.54}$$

to calculate an expression for $\delta^{(2)}$. After algebraic manipulation, we find that this expression can be written in the following form:

$$\delta^{(2)} = J_1^{\text{2PPT}}(\tau)\varphi_0^2 + J_2(\tau)\boldsymbol{\Theta}_0 + J_3^{\text{2PPT}}(\tau)(\nabla\varphi_0)^2 + J_4(\tau)\mathcal{F} + J_5^{\text{2PPT}}(\tau)\varphi_0\nabla^2\varphi_0 \, , \tag{8.55}$$

where $J_2$ and $J_4$ remain unaltered from their forms given in Chapter 4, and $J_1^{\text{2PPT}}$, $J_3^{\text{2PPT}}$ and $J_5^{\text{2PPT}}$ are given by

$$J_1^{\text{2PPT}} = \Bigg[ -\frac{4\mathcal{H}\mathcal{D}'}{\mathcal{H}_0^2\Omega_{m0}} g_{in}\left( -\frac{5}{3}(a_{\text{NL}} - 1) - 1 \right) - \frac{8\mathcal{H}^2\mathcal{D}^2}{a\mathcal{H}_0^2\Omega_{m0}} - \frac{2\mathcal{H}}{\mathcal{H}_0^2\Omega_{m0}}(b_1^{\text{2PPT}})' \Bigg] \, , \tag{8.56}$$

$$J_3^{\text{2PPT}} = \Bigg[ -\frac{4}{9}\frac{\mathcal{D}g_{in}}{\mathcal{H}_0^2\Omega_{m0}}(1 + 10a_{\text{NL}}) + \frac{10}{3}\frac{\mathcal{D}g}{\mathcal{H}_0^2\Omega_{m0}} - \frac{8(\mathcal{D}')^2}{9\mathcal{H}_0^4\Omega_{m0}^2} + \frac{4}{\mathcal{H}_0^2\Omega_{m0}} b_1^{\text{2PPT}} \Bigg] \, , \tag{8.57}$$

$$J_5^{\text{2PPT}} = \Bigg[ \frac{8\mathcal{D}g_{in}}{3\mathcal{H}_0^2\Omega_{m0}}\left( -\frac{5}{3}(a_{\text{NL}} - 1) - 1 \right) + \frac{16}{3}\frac{\mathcal{D}g}{\mathcal{H}_0^2\Omega_{m0}} + \frac{4}{\mathcal{H}_0^2\Omega_{m0}} b_1^{\text{2PPT}} \Bigg] \, , \tag{8.58}$$

where $b_1^{\text{2PPT}}(a)$ is calculated by numerical integration. The functions $J_1^{\text{2PPT}}$, $J_3^{\text{2PPT}}$ and $J_5^{\text{2PPT}}$ are plotted in Figures 8.2, 8.3 and 8.4.

It is important to note that in Figure 8.2, the functions do not approach their precise Einstein-de Sitter values of $(J_1)^{(\text{EdS})} = 4$ and $(J_1^{\text{2PPT}})^{(\text{EdS})} = -4$ at high redshift, even though the universe is well approximated by Einstein-de Sitter at this point since $\Omega_\Lambda \to 0$. This is because the value of $g_{in}$ in ΛCDM cosmologies is fixed, whilst in the simpler Einstein-de Sitter case, it is just normalised to 1, since $g_{\text{EdS}} = 1$ at all times. The implication is that the difference is purely one of normalisation and is nothing to be concerned about. The key thing to observe is that the Einstein-de Sitter *behaviour* is recovered at high redshift (namely $J_1 \to$ const). It has been verified that the precise Einstein-de Sitter limits of all these functions are recovered numerically if the calculations are run with $\Omega_{m0} \to 1$.

It can be seen in Figure 8.3 that the magnitude of the $J_3^{\text{2PPT}}$ function is reduced from that of the $J_3$ function at low redshift. Figure 8.4 shows that the $J_5^{\text{2PPT}}$ function

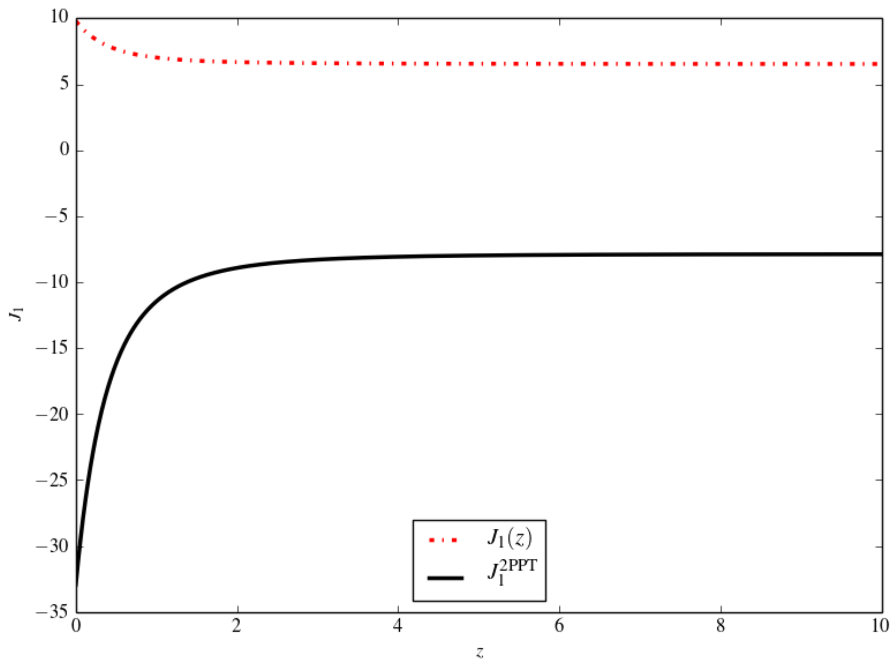


Figure 8.2.: A plot of the $J_1^{2PPT}(z)$ solution, compared to the $J_1(z)$ solution from second order perturbation theory. The values of both functions are calculated via the numerical integration method given above.

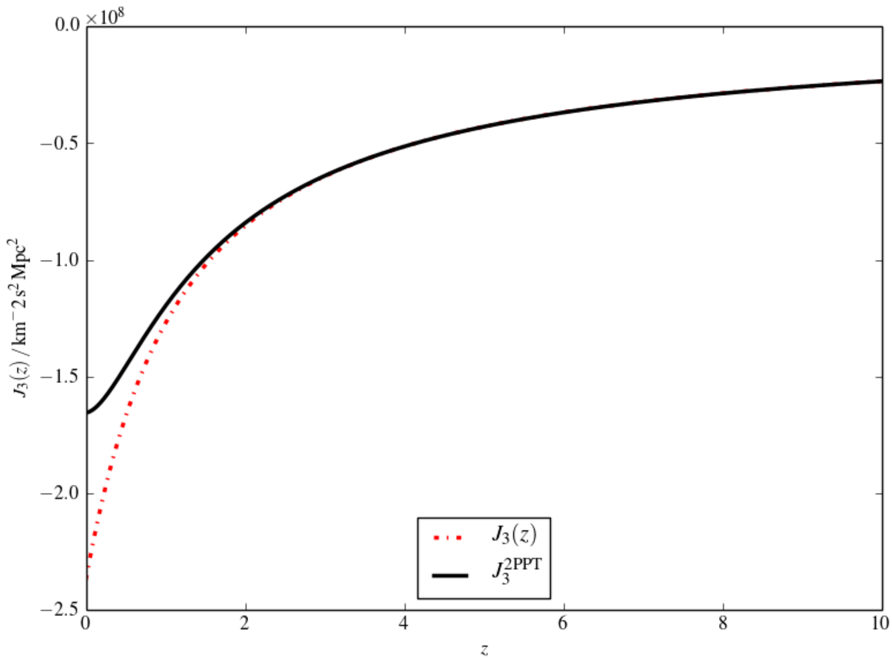


Figure 8.3.: A plot of the $J_3^{2PPT}(z)$ solution, compared to the $J_3(z)$ solution from second order perturbation theory. The values of both functions are calculated via the numerical integration method given above.

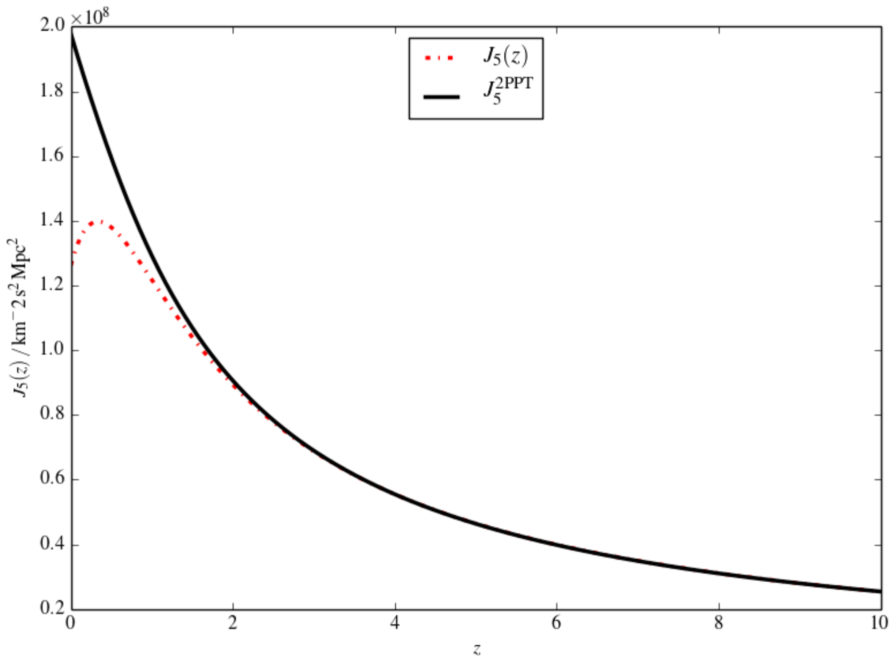


Figure 8.4.: A plot of the $J_5^{\rm 2PPT}(z)$ solution, compared to the $J_5(z)$ solution from second order perturbation theory. The values of both functions are calculated via the numerical integration method given above.

is larger than the $J_5$ function at low redshift. These changes measure the difference in the solution for $\delta^{(2)} + \delta_{\rm N}^{(2)}$ compared to $\delta_2$ from cosmological perturbation theory.

The expressions for the 2PPT kernel correction functions remain unchanged from those presented in Chapter 4, except that $J_1$, $J_3$ and $J_5$ are replaced by their 2PPT versions. We repeat the form of the kernels here for the reader's convenience:

$$\alpha^{\rm 2PPT}(k,\tau) = \left(1 - \frac{\mathcal{F}}{\mathcal{D}^2}\right) + \frac{(\mathcal{H}_0^2\Omega_{m0})^2}{\mathcal{D}^2}\left[-\frac{9(4J_3^{\rm 2PPT} + J_4)}{8k^2} + \frac{9J_1^{\rm 2PPT}}{k^4} - \frac{3J_2}{8k^4}\right], \tag{8.59}$$

$$\beta^{\rm 2PPT}(k,\tau) = 2 + \frac{(\mathcal{H}_0^2\Omega_{m0})^2}{\mathcal{D}^2}\left[-\frac{9(J_3^{\rm 2PPT} + J_5^{\rm 2PPT})}{2k^2} + \frac{18J_1}{k^4} - \frac{3J_2}{2k^4}\right], \tag{8.60}$$

$$\gamma^{\rm 2PPT}(k,\tau) = \frac{(\mathcal{H}_0^2\Omega_{m0})^2}{\mathcal{D}^2}\left[-\frac{9J_5^{\rm 2PPT}}{8k^2} + \frac{9J_1^{\rm 2PPT}}{4k^4}\right]. \tag{8.61}$$

These functions can be inserted into the Fourier kernel,

$$\mathcal{K}_2^{\rm 2PPT}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{(\beta-\alpha) + \frac{\beta}{2}\,\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \alpha\left(\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2\right)^2 + \gamma\left(\frac{k_1}{k_2}-\frac{k_2}{k_1}\right)^2}{\left(1+\frac{3\mathcal{H}^2 f}{k_1^2}\right)\left(1+\frac{3\mathcal{H}^2 f}{k_2^2}\right)}\,, \tag{8.62}$$

where the superscript "2PPT" is suppressed in order to save space. This enables the calculation of the correlation functions of the dark matter overdensity, in the same way that was done in Chapters 4 and 7.

# 8.6. Leading order bispectra

In this section, we will present the results of the leading order corrections to the bispectra of the metric potential $\psi^{(2)}+U^{(2)}$ and the dark-matter overdensity $\delta^{(2)}+\delta_{\rm N}^{(2)}$.

## 8.6.1. Gravitational potentials

The leading order contribution to the dimensionless bispectrum of the scalar metric potential, $\psi+U$, is calculated in the usual fashion:

$$\begin{aligned}&\langle(\psi+U)(\psi+U)(\psi+U)\rangle \sim \langle(\psi^{(2)}+U^{(2)})(\psi^{(1)}+U^{(1)})(\psi^{(1)}+U^{(1)})\rangle\\ &= (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\mathcal{K}_2^{\psi+U}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k})P^{\psi}(q_1)P^{\psi}(q_2) + \text{ 2 cycl. perms}\,,\end{aligned} \tag{8.63}$$

where $P^{\psi}(k)$ is the dimensional linear power spectrum of the gravitational potential, predicted by *CLASS*. The implication of this result is that we can write the dimensionless bispectrum, $B^{\psi+U}$, in terms of the kernel. In terms of the dimensionless linear power spectrum of gravitational potentials, $\Delta^{\psi}=\frac{2\pi^2}{k^3}P^{\psi}(k)$, the leading order bispectrum can be written as

$$B^{\psi+U}(k_1,k_2,k_3) = \mathcal{K}_2^{\psi+U}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k})\Delta^{\psi}(q_1)\Delta^{\psi}(q_2) + \text{ 2 cycl. perms}\,. \tag{8.64}$$

The corresponding result in cosmological perturbation theory is calculated using the same process used to calculate the dark matter bispectrum, outlined in Chapter 4,

$$B^{\Psi}(k_1,k_2,k_3) = \mathcal{K}_2^{\Psi}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k})\Delta^{\psi}(q_1)\Delta^{\psi}(q_2) + \text{ 2 cycl. perms}\,. \tag{8.65}$$

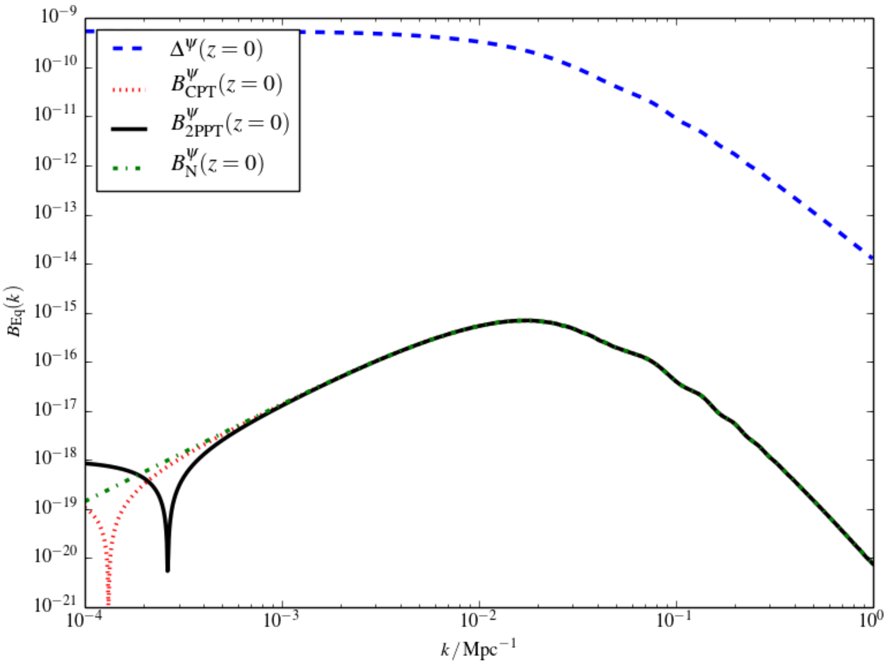


Figure 8.5.: A plot of the equilateral configuration of the bispectrum of gravitational potentials in second-order cosmological perturbation theory vs the 2PPT version of this quantity, evaluated at redshift $z = 0$.

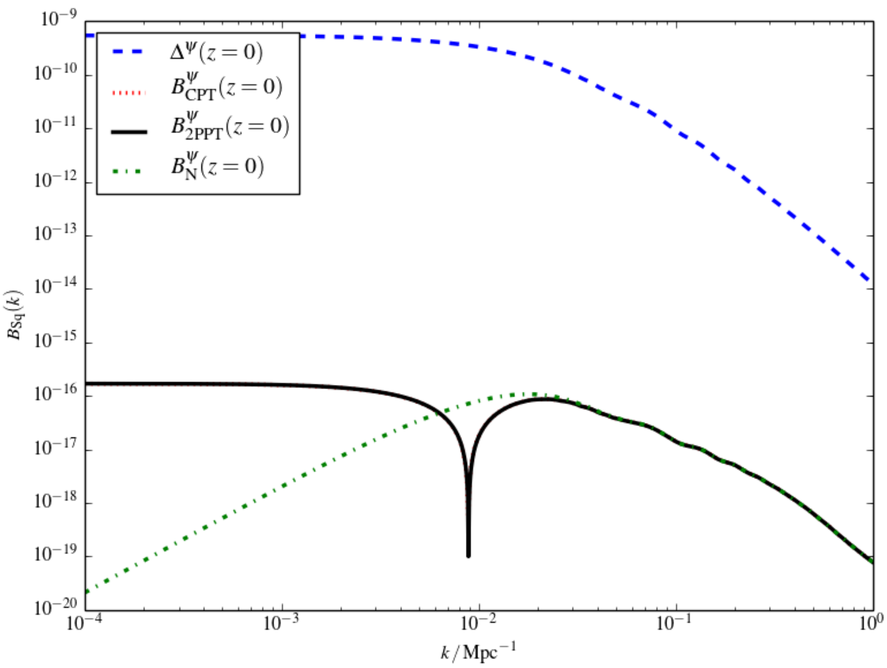


Figure 8.6.: A plot of the squeezed configuration of the bispectrum of gravitational potentials in second-order cosmological perturbation theory vs the 2PPT version of this quantity, evaluated at redshift $z = 0$.

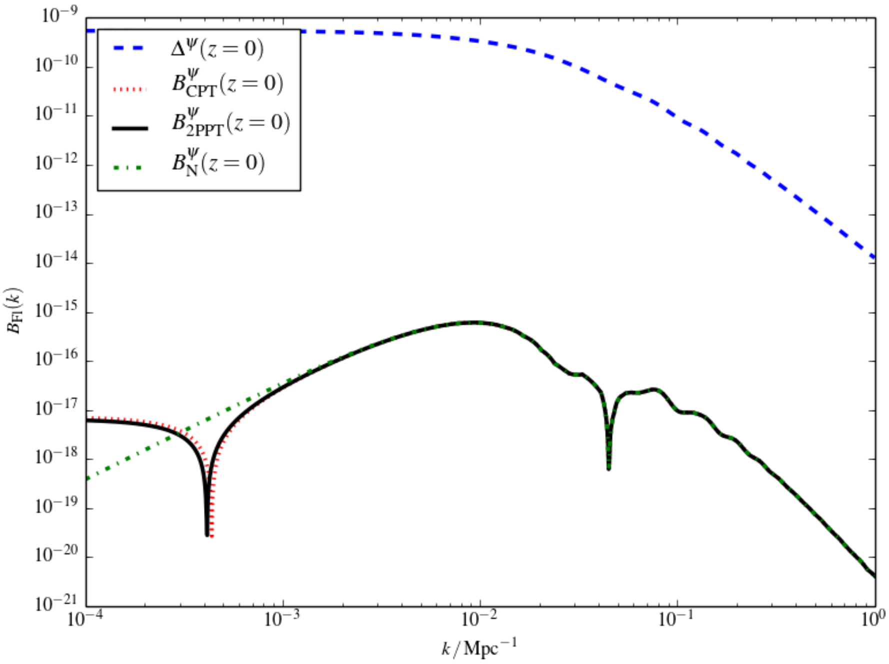


Figure 8.7.: A plot of the flattened configuration of the bispectrum of gravitational potentials in second-order cosmological perturbation theory vs the 2PPT version of this quantity, evaluated at redshift $z = 0$.

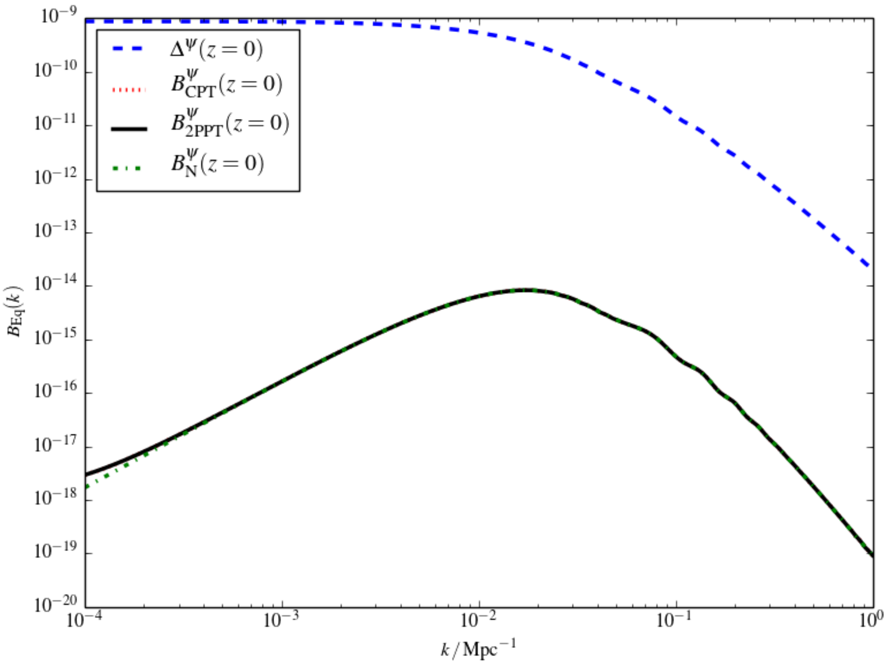


Figure 8.8.: A plot of the equilateral configuration of the bispectrum of gravitational potentials in second-order cosmological perturbation theory vs the 2PPT version of this quantity, evaluated at redshift $z = 5$.

We plot the dimensionless bispectrum of gravitational potentials at redshift $z = 0$ in various configurations in Figures 8.5, 8.6 and 8.7. The results are plotted together with the linear power spectrum $\Delta^{\psi}$ (black line) in order to give the reader a reference for the magnitudes of the quantities involved. These plots demonstrate the deviation of the 2PPT solution for the gravitational potential at small values of $k$. This behaviour can be anticipated from the form of the integral kernel; Newtonian terms come with an extra $k^2$ and hence dominate at large $k$, whereas relativistic corrections are dimensionless, and hence are only visible at small values of $k$. The deviation from the cosmological perturbation theory solution is maximised in the equilateral configuration, whilst it is significantly reduced in the flattened case.

In the case of the squeezed limit, Figure 8.6, it is clear that the differences between the cosmological perturbation theory solution and the 2PPT solution is negligible even at scales of $k \sim 10^{-4}$. This is because the kernel is dominated by the terms proportional to $\hat{\boldsymbol{q}}_1 \cdot \hat{\boldsymbol{q}}_2$ in this limit, and these terms are the same in both the cosmological perturbation theory and 2PPT case. Agreement with the cosmological perturbation theory result is at the sub-percent level on even the largest scales.

In Figure 8.8, we plot the equilateral bispectrum of the gravitational potential at redshift $z = 5$. It is clear that there is no noticeable deviation from the second-order perturbation theory result. This is to be expected as $z = 5$ is already well within the matter-dominated era, where the gravitational potentials in second-order perturbation theory and 2PPT are the same.

### 8.6.2. Dark matter overdensity

The second approximation to the dark-matter overdensity leads to the following expression for the leading order 2PPT bispectrum:

$$B^{\rm 2PPT}(k_1, k_2, k_3) = \mathcal{K}_2^{\rm 2PPT}(\boldsymbol{k}_1, \boldsymbol{k}_2) P(k_1) P(k_2) + \text{ cycl perms} . \tag{8.66}$$

This quantity is plotted at redshift $z = 0$ in Figures 8.9, 8.10 and 8.11, in the equilateral, squeezed and flattened configurations respectively.

Figures 8.9 8.10 and 8.11 display more significant deviation from the results of cosmological perturbation theory than in the Einstein-de Sitter case examined in Chapter 7. This is to be expected, since the differences in the field equations are more significant. In the Einstein-de Sitter case, time derivatives of the first order solution vanished, meaning that the omission of time-dependent terms due to the 2PPT counting scheme was not a factor causing deviation. However in the ΛCDM

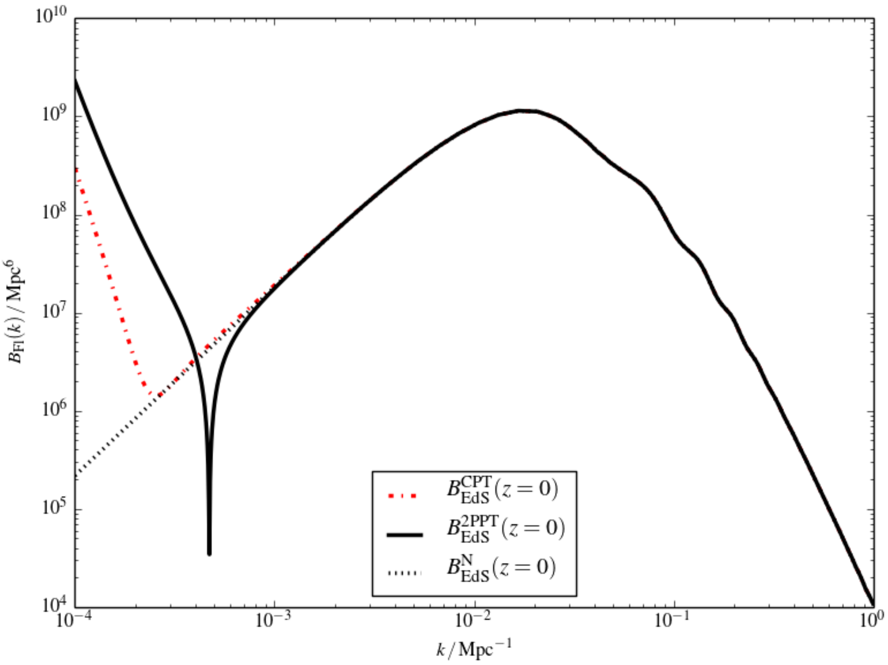


Figure 8.9.: A plot of the equilateral configuration of the bispectrum of dark matter in second-order cosmological perturbation theory vs the 2PPT version of this quantity, evaluated at redshift $z = 0$.

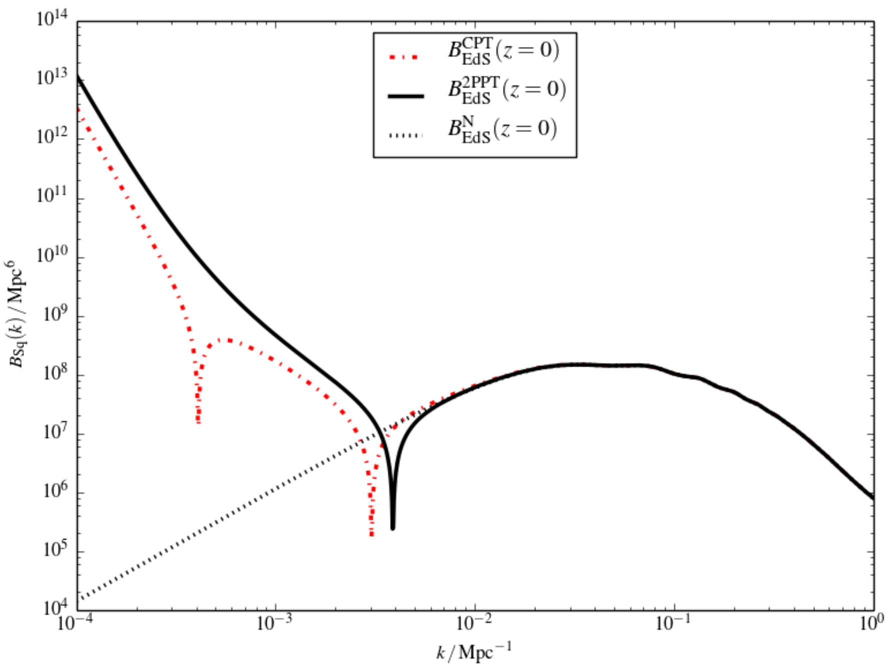


Figure 8.10.: A plot of the squeezed configuration of the bispectrum of dark matter in second-order cosmological perturbation theory vs the 2PPT version of this quantity, evaluated at redshift $z = 0$.

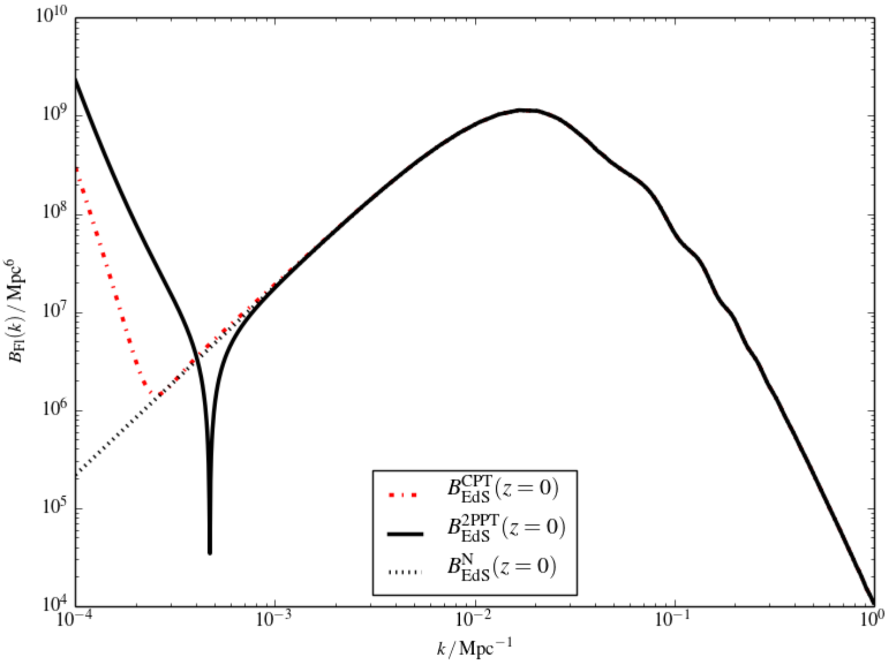


Figure 8.11.: A plot of the flattened configuration of the bispectrum of dark matter in second-order cosmological perturbation theory vs the 2PPT version of this quantity, evaluated at redshift $z = 0$.

case, time derivative terms like $\varphi'^2$ (and terms involving the cosmological constant like $a^2\mathcal{H}^2\varphi^2$) are present in the second-order perturbation theory field equations, but not the 2PPT field equations, leading to further, more significant deviations.

## 8.7. Discussion

How should we interpret the results shown in Figures 8.5 - 8.11? As emphasised in Chapter 7, conclusions about the merit of the entire 2PPT scheme should not be based on proximity to the results of second order cosmological perturbation theory. As stated before, second-order perturbation theory is a theory of second order corrections to the Einstein field equations in a regime where all fluctuations can be thought of as small with respect to the background. The unperturbed two-parameter field equations instead describe the evolution of *first-order* perturbations on top of a background which itself contains full nonlinear Newtonian structure, on length scales less than $L_N \sim 100$ Mpc. As such, the two-parameter field equations *do not* contain any description of gravitational self-interaction on the largest scales. In order to include such physics within the two-parameter expansion, one could in principle go to second-order in cosmological perturbations.

The 2PPT scheme identifies terms that might be thought of as traditionally higher-order in perturbation theory that are enhanced by the presence of nonlinear structure on small scales. Since some, but not all, of the second-order (and third-order!) perturbation theory terms are brought down into the first-order equations, the *fundamental structure of the governing equations is changed*. We choose a minimal approach to approximating solutions to these equations, by approximating the physics on short length scales using Newtonian perturbation theory. This approach, whilst sacrificing the potential to include full nonlinearity, enables approximations to solutions to be constructed without too much difficulty, and importantly, serves to highlight the differences that the different structures of the equations cause in the solutions for the perturbations.

By plotting the bispectrum, a statistic that is *identically zero in first order perturbation theory* (with Gaussian initial conditions), we draw attention to the fact that quantities that might be traditionally thought of as higher-order may have to be considered simultaneously with first-order perturbations. In particular, were the Newtonian quantities to be allowed to remain fully nonlinear, one might anticipate that there would significant departures from the predictions of second-order perturbation theory on *all scales*, not just in the non-perturbative regime. In particular, this could potentially happen because of the presence of enhanced source terms in the first-order equations. In second-order perturbation theory, quantities derived by finding the split, $\Psi_2 - \Phi_2$, can directly cancel with terms in the field equations. This cancellation no longer happens in the 2PPT approach, where we use Newtonian perturbation theory to approximate the short-scale solutions. In a universe with true nonlinear structure present, one would presumably expect this cancellation to fail generically as well, leading to significant differences in the solutions for perturbations on the largest scales.

We find that differences between the 2PPT scheme and cosmological perturbation theory are much more significant in the ΛCDM universe than in the Einstein-de Sitter case. The reason for this, as stated before, is that the first order gravitational potential is time-dependent in ΛCDM, whereas it is time-independent in Einstein-de Sitter. When the first-order gravitational potential is time-independent, a number of terms in the constraint equations which are different in cosmological perturbation theory and 2PPT vanish identically, bringing the two schemes significantly more into alignment with each other. This is somewhat qualitatively reflected in the fact that the 2PPT ΛCDM dark matter bispectra look significantly closer to their Einstein-de Sitter counterparts than those in cosmological perturbation theory. Furthermore, the evolution of the gravitational potential is affected on the largest scales in the

ΛCDM universe, whereas in the Einstein-de Sitter case the evolution of the scalar metric degree of freedom remains unaffected.

Whilst we hypothesised that one potentially useful application of the 2PPT scheme might be to calculate a third approximation to $\delta_{\rm 2PPT}$, and therefore, to $P_{1-\rm loop}(k)$, the outcome of the investigation into 2PPT in the ΛCDM universe seems to indicate that this is unlikely to be a fruitful direction to continue in. The reasons for this conclusion are two-fold. Firstly, in Einstein-de Sitter, the time dependency of the source terms of the evolution equation for the gravitational potential is very simple, with all terms being proportional to the scale factor $a(\tau)$. One consequence of this is that only one differential equation must be solved to calculate the evolution. When performing a third approximation, it can be inferred that the only possible time dependencies that source terms can have are either $\propto a(\tau)^2$, $\propto a(\tau)$ or $\propto$ const, thus implying that a maximum of three differential equations would need to be solved, no matter how hypothetically complicated the spatial dependence might be. In ΛCDM however, each individual term has its own unique time dependence, implying that each contribution to the source term must be solved for separately. Since the number of source terms in a third approximation is expected to be large, this quickly nullifies any potential calculational advantage that 2PPT might have to offer.

Secondly, as demonstrated by Figures 8.9, 8.10 and 8.11, the deviations from Newtonian behaviour in ΛCDM 2PPT even have a different sign to the relativistic corrections in second-order perturbation theory. This means that any hope of using 2PPT as a "shortcut" to access results from traditional relativistic perturbation theory should probably be abandoned. Instead, the outcome of this study should perhaps be interpreted as a warning to those using perturbation theory. In the real universe, where nonlinear structure is present on small scales, the structure of the perturbative hierarchy of the Einstein field equations is liable to change, as some higher order terms may be larger than expected. Consequentially, one must be extremely careful when interpreting the results of perturbation theory in the real universe, even on the largest scales.

# 9. Conclusions

In this thesis we have examined two different perturbative approaches to modelling large scale structure in the late universe, considering the advantages and disadvantages of each, before proceeding to investigate a new formalism which attempts to combine aspects of both type of expansion. In Chapter 5, we considered the structure of gauge transformations in both cosmological perturbation theory (applicable on large scales) and post-Newtonian perturbation theory (applicable on small scales). While both treatments of gravitational fields have their own well-defined gauge problems, we find that most of the particular gauge choices that are used in cosmology are not valid using post-Newtonian theory in the presence of non-linear structures. In particular, the spatially flat gauge, the synchronous gauge, the comoving orthogonal gauge, the total matter gauge, the N-body gauge, and the uniform density gauge are all beyond the limits of what it is possible to achieve by applying an infinitesimal coordinate transformation in the post-Newtonian sector.

In contrast, the longitudinal gauge and the Newtonian motion gauge both appear to be well-defined in the post-Newtonian treatment of gravitational fields, as well as in cosmological perturbation theory. The former of these is a very simple choice of gauge, and is already well-known to give sensible results when extrapolating the cosmological perturbation theory to the regime of non-linear density contrasts. Here we formalise this result, and explain its veracity, by showing it can be realised in post-Newtonian expansions (which are purposefully constructed to model weak-field gravity in such scenarios). The latter gauge choice (Newtonian motion gauge) requires numerical integration of a non-local differential equation in order to be applied in practice. If this is possible, then it should allow one to post-process cosmological Newtonian N-body simulations in order to derive relativistic corrections to gravitational fields, and to determine the effects of these fields on observables without having to perform additional simulations. This is an intriguing possibility, which it would be interesting to apply in practice, and would enable the calculation of relativistic corrections in situations with contributions from radiation and neutrinos.

These results provide support for the use of longitudinal gauge in studies that attempt to simultaneously model both small-scale non-linear structures as well as linear structures on large scales, e.g the numerical code *gevolution* or the two-parameter perturbative approach. On the other hand, they also provide a warning that other choices of gauge should be applied with care. In particular, the fact that one cannot use gauge transformations to realise synchronous coordinates in post-Newtonian theory has potentially interesting consequences. While this result does not imply that it is impossible in general to find a coordinate system where the time coordinate corresponds to the proper time of observers comoving with matter, it does means that the difference between a synchronous coordinate system and the coordinates of a perturbed FLRW space-time cannot be related by a small gauge generator. That is, the difference between these two different notions of time is large, in the sense defined by the perturbative expansion, and is therefore unattainable by small gauge transformations. Such a result would appear to have significance for a number of studies that use proper time in the presence of nonlinear structures, such as the calculation of galaxy bias on hypersurfaces of constant proper time. It may also go someway to explaining the vastly different expectations that different groups of cosmologists appear to have when considering the problem of cosmological backreaction.

In Chapter 6 we extend the two-parameter perturbation expansion first proposed in [125] to include background radiation and a cosmological constant. In doing so we use both cosmological perturbation theory and the post-Newtonian expansion. As this expansion is able to model large density contrasts and different matter components it therefore both contains the essential features of the real Universe and has a number of potential advantages over standard cosmological perturbation theory. We derived the two-parameter perturbed field equations valid for structure on the order of a fraction of the horizon size, the two-parameter gauge transformations of the matter sector, and construct gauge-invariant quantities in this sector. The consistency of the gauge transformations requires not only gravitational potentials and matter perturbations at the orders expected from post-Newtonian gravity and cosmological perturbation theory alone, but also a number of others at orders in perturbation which may not naively have been expected. We have therefore identified a minimal set of perturbations that are required for mathematical consistency of the problem, and written down gauge-invariant versions of the field equations that contain them all.

We find that the small-scale Newton-Poisson equation for the scalar gravitational potential occurs at the same order in perturbations as the Friedmann equation, but that they can be separated after the introduction of a suitable homogeneity scale. At leading order, this results in a small-scale Newton-Poisson equation sourced by the inhomogeneous part of the Newtonian energy density, and large-scale Friedmann equations sourced by the spatial average of the leading-order parts of the energy density, pressure, and the cosmological constant. Our results give no indications that the effects of small-scale nonlinearities should be expected to cause acceleration of the large-scale Universe, but we do find that they should be expected to affect large-scale perturbations. This is because the higher-order field equations include quadratic Newtonian potentials within the effective fluid terms, but also terms that couple long wavelength first order cosmological perturbations to leading order Newtonian quantities. Within this prescription we observe a mixing of scales, as well as mode-mixing at what would normally be considered to be linear order in cosmological potentials.

By presenting the higher-order field equations in terms of an effective fluid we are able to highlight the similarities and differences between our formalism and regular cosmological perturbation theory. We expect this to aid further application of our equations by allowing some standard techniques from cosmological perturbation theory to be imported. Our effective fluid description also enables an easier physical interpretation of the effects of nonlinearities in the field equations, which clearly lead to (for example) a large-scale effective pressure and anisotropic stress.

We have derived and presented the relativistic Euler equations that exist in the two-parameter expansion. These equations describe the evolution of density perturbations and peculiar velocities for a self-gravitating perfect fluid in an FLRW background. These equations are written down in gauge-invariant variables, and were used to confirm that the constraint equations from the two-parameter perturbation expansion are consistently evolved, despite the fact that terms can change order under differentiation. This gives confidence that the scheme is internally self-consistent and complete, and can be used to model the relativistic effects of non-linear structures in perturbation theory. The resulting Euler equations for the inhomogeneous part of the leading-order matter density and the peculiar velocity, together with the leading order gravitational Poisson equation, reproduce the standard results of Newtonian perturbation theory on an expanding background, as long as cosmological contributions to the peculiar velocity are included. These leading-order equations have well-known solutions in terms of Green's functions and numerical N-body simulations. Subsequent higher-order equations that govern the leading-order contributions to the large-scale gravitational potentials are then given as linear partial differential equations that contain the known solutions to the lower-order Newtonian equations as source terms.

In Chapter 7, we have presented a blueprint for finding analytic approximations to a simplified version of the two-parameter equations in an Einstein-de Sitter universe. By taking the well-known formal solutions from Newtonian perturbation theory in Einstein-de Sitter universes, inserting them into the 2PPT equations, and assuming a corresponding perturbative expansion in the quantities in the 2PPT equations themselves, we are able to perturbatively construct order-by-order approximations to the full 2PPT dynamics, in analogy to the way that regular perturbation theory constructs order-by-order approximations to the dynamics described by the fully nonlinear Einstein equations. It is shown that the evolving part of the second-order solution is described completely by the second-order NPT result, and that any second-order relativistic corrections arise only in the form of initial conditions.

Providing the Newtonian equations can be linearised, we find that the second approximations to solutions of the 2PPT equations are very similar to those of standard second-order cosmological perturbation theory, with difference only arising on ultra-large scales (due to the different structure of the field equations). To highlight the differences between each approach, we have focused on the tree-level bispectrum, one of the simplest statistics to calculate in perturbation theory. We find reasonable agreement for scales with $k > 10^{-3}$ between the second approximation to 2PPT and second-order CPT, indicating that 2PPT does well at approximating a universe with exclusively linear fluctuations at these scales. Differences arise at ultra-large scales due to the fact that we have only considered 2PPT large-scale fluctuations up to first order in $\epsilon$.

In Chapter 8, we extend the method for approximating solutions using Newtonian perturbation theory to the case of a background universe with a cosmological constant. Following the same methodology as in the Einstein-de Sitter universe, we find significantly greater differences in the relativistic corrections predicted by the 2PPT scheme compared to cosmological perturbations in the ΛCDM universe. This difference is explained by the fact that the first approximation to the scalar gravitational potential becomes time-dependent in the ΛCDM universe - leading to a larger number of terms in the cosmological perturbation theory field equations that do not have counterparts in 2PPT. Whilst these results indicate that any hopes of using 2PPT as a "shortcut" to access results from higher-order perturbation theory on intermediate scales are not likely to produce reliable results, they do indicate that fundamental differences in the structure of the gravitational field equations due to short-scale nonlinear structures could indeed have significant consequences for the calculation of relativistic effects at large-scales, which may lead to interesting projects in the future.

## 9.1. Future work

The two-parameter formalism on which this thesis is based is in its infancy. We have presented a roadmap for simple perturbative calculations using approximate solutions from Newtonian perturbation theory for the leading order dynamics; however, the long term objective must be to keep the leading order Newtonian dynamics fully nonlinear and investigate the effect on the corresponding long-wavelength cosmological fluctuations. Since the effective fluid terms are all constructed from the solution to the short-scale Newtonian gravitational potential, their properties should be able to be determined from N-body simulations [39, 138]. Once the properties of these effective fluids have been identified, one can proceed to solve the cosmological equations for the long-wavelength perturbations. This method of solution is available to us because of the hierarchical nature of the perturbation equations; short-scale fluctuations appear at lower-order compared to cosmological perturbations, and so can be solved for before cosmological perturbations.

The calculations given in Chapters 7 and 8 neglect the existence of post-Newtonian relativistic corrections on small scales. The justification for this is that by using Newtonian perturbation theory, we assume to some extent that the density contrast must be quasilinear [84], in turn meaning that post-Newtonian corrections are liable to be smaller than in the fully nonlinear case. However, much of the power of the two-parameter approach resides in the fact that it is able to consistently model both long-wavelength cosmological perturbations and post-Newtonian relativistic corrections. Indeed, the formalism seems to indicate that in a universe with nonlinear structure on small scales, one should expect post-Newtonian relativistic corrections to appear in the same field equations as cosmological perturbations, so the possibility of interplay must not be overlooked. The potential existence of the "mixed" terms described in Chapter 6 only serves to accentuate this point. One potential interpretation of the occurrence of such terms is as additional source terms in a post-Newtonian expansion, indicating the existence of source terms at orders in which they do not usually appear in the typical expansion. Such terms could indicate the presence of interplay between scales. Such possibilities indicate that this formalism is worthy of further investigation.

The two-parameter formalism is well-suited for application to a number of problems of relevance for galaxy surveys. In particular, a treatment of the subject of galaxy bias in the two-parameter formalism would be desirable [123, 139], due to the applicability of the post-Newtonian expansion on small scales. Following a treatment of the bias, one could progress to a treatment of observables like the $n$-point statistics of galaxy number counts [107, 108, 110, 140–145].

Applications in the area of weak lensing are also foreseeable, since large and small scales are typically coupled due to the integrated effect of inhomogeneities along the photon path [146]. Relativistic corrections to the convergence and shear are to be expected in the correlation between two galaxies that are separated by large distances, yet if both galaxies are aligned with respect to the line of sight, the correlation will also be strongly affected by short-scale nonlinear effects. Any nonlinear structure close to the observer can also be expected to have a strong effect on correlations between even widely separated galaxies, since all photon trajectories converge on the observer. As such, the analysis of large-scale weak lensing data fundamentally requires modelling of relativistic effects and nonlinear effects simultaneously - exactly the type of physical phenomenon suited to modelling with the two-parameter expansion. Stage IV Weak lensing surveys will require models able to probe nonlinear scales where baryonic effects become important. Although two-parameter perturbation theory in the form presented here is not able to deal with such effects, there are interesting possibilities for modification of the Newtonian sector of the theory to incorporate some of the baryonic effects. For example, prescriptions already exist to modify Newtonian N-body simulations to incorporate baryonic effects via a process called "baryonification" [147]. It is conceivable that the Newtonian solutions obtained from such simulations could be extended to larger scales using the two-parameter approach. This in turn might enable the simultaneous modelling of relativistic and baryonic effects.

The two-parameter perturbation theory set up can also be feasibly extended to modified gravity scenarios by simply performing a two-parameter expansion on the quantities in modified field equations in exactly the same way as is done in the general relativity case. It may also be possible to construct a parameterised version of the full two-parameter perturbation theory. The short-scale post-Newtonian corrections can be parameterised in the standard fashion following the parameterised post-Newtonian (PPN) scheme [50]. Parameterisations of the linear modifications to the Einstein equations already exists in the literature (see for example, [148–151]) - it is anticipated that these parameterisations should be recovered in the limit where collapsed nonlinear structures are not present [52, 54]. Combining this parameterisation with a PPN approach to the short-scale physics may enable these large scale parameters to be connected with the well known PPN parameters.

In conclusion, we hope that two-parameter perturbation theory provides a new arena in which questions about the nonlinearity of general relativity and its effect on large-scale structure can be effectively investigated. The approaches presented here enable researchers to start using two-parameter perturbation theory to perform practical calculations, analogous to those that have already been performed using more traditional methods. We hope that the methodology may also influence those working on similar problems using different approaches, such as the weak-field approximation to general relativity, and that the formal derivations of two-parameter perturbation theory may highlight more clearly the assumptions that go into such schemes. The era of precision cosmology is only in its infancy, and new techniques and approaches will become increasingly necessary as the data from next-generation surveys starts to be collected. We hope that two-parameter perturbation theory can find its place amongst the variety of new techniques that will be used.

# A. Two-parameter perturbed Ricci and stress-energy tensors

This appendix provides component expressions for the perturbed Ricci tensor and the perturbed energy-momentum tensor. These expressions are subsequently used to derive the gravitational field equations in Appendix C. The expressions for the perturbed Ricci tensor were first calculated in [125]. We make no claim that they comprise an original component of this thesis, merely repeating them here so as to provide the reader with a self-consistent presentation of the full two-parameter perturbation theory. We make no assumptions about the relative magnitude of $\epsilon$ and $\eta$ in this appendix, nor do we assume anything about the length scales $L_C$ and $L_N$.

Firstly, expanding the components of the Ricci tensor in both the $\epsilon$ and $\eta$ parameters, we find that the non-vanishing contributions to each component are given by:

$$\begin{aligned} R_{00} &= R_{00}^{(0,0)} + R_{00}^{(0,2)} + R_{00}^{(0,3)} \\ &\quad + \tfrac{1}{2}R_{00}^{(0,4)} + R_{00}^{(1,0)} + R_{00}^{(1,1)} + R_{00}^{(1,2)} + \dots \end{aligned} \tag{A.1}$$

$$R_{0i} = R_{0i}^{(0,2)} + R_{0i}^{(0,3)} + R_{0i}^{(1,0)} + R_{0i}^{(1,2)} + \dots \tag{A.2}$$

$$\begin{aligned} R_{ij} &= R_{ij}^{(0,0)} + R_{ij}^{(0,2)} + R_{ij}^{(0,3)} \\ &\quad + \tfrac{1}{2}R_{ij}^{(0,4)} + R_{ij}^{(1,0)} + R_{ij}^{(1,1)} + R_{ij}^{(1,2)} + \dots . \end{aligned} \tag{A.3}$$

Ellipses denote higher-order terms, which will not be required.

Each term in these equations has an order of smallness in $\epsilon$ and $\eta$, as indicated by the superscript in brackets, and a length scale given by $L_N^{-2}$, $L_C^{-2}$ or $L_C^{-1}L_N^{-1}$. We have not indicated this directly on each of the terms in the expansion, but it is important to keep track of these length scales as subsequent calculations will require us to convert $L_C$ into $L_N$ in order to write consistent field equations in a single system of units.

The constituent terms on right-hand side of Eq. (A.1) are given by

$$R_{00}^{(0,0)} = -3\frac{\ddot{a}}{a} \sim \frac{1}{L_C^2} \tag{A.4}$$

$$R_{00}^{(0,2)} = -\frac{1}{2a^2}h_{00,ii}^{(0,2)} \sim \frac{\eta^2}{L_N^2} \tag{A.5}$$

$$R_{00}^{(0,3)} = \frac{\dot{a}}{a^2}h_{0i,i}^{(0,3)} - \frac{\dot{a}}{a}h_{ii,0}^{(0,2)} - \frac{3\dot{a}}{2a}h_{00,0}^{(0,2)} \sim \frac{\eta^3}{L_C L_N} \tag{A.6}$$

$$\begin{aligned} R_{00}^{(0,4)} = & -\frac{1}{2a^2}\left(h_{00,i}^{(0,2)}\right)^2 - \frac{1}{2a^2}h_{00,ii}^{(0,4)} - h_{ii,00}^{(0,2)} \\ & + \frac{2}{a}h_{0i,0i}^{(0,3)} + \frac{1}{2a^2}h_{00,i}^{(0,2)}\left(2h_{ij,j}^{(0,2)} - h_{jj,i}^{(0,2)}\right) \\ & + \frac{1}{a^2}h_{00,ij}^{(0,2)}h_{ij}^{(0,2)} \sim \frac{\eta^4}{L_N^2} \end{aligned} \tag{A.7}$$

$$\begin{aligned} R_{00}^{(1,0)} = & -\frac{1}{2a^2}h_{00,ii}^{(1,0)} - \frac{1}{2}h_{ii,00}^{(1,0)} + \frac{\dot{a}}{a^2}h_{0i,i}^{(1,0)} \\ & - \frac{\dot{a}}{a}h_{ii,0}^{(1,0)} + \frac{1}{a}h_{0i,0i}^{(1,0)} - \frac{3\dot{a}}{2a}h_{00,0}^{(1,0)} \sim \frac{\epsilon}{L_C^2} \end{aligned} \tag{A.8}$$

$$R_{00}^{(1,1)} = -\frac{1}{2a^2}h_{00,ii}^{(1,1)} \sim \frac{\epsilon\eta}{L_N^2} \tag{A.9}$$

$$\begin{aligned} R_{00}^{(1,2)} = & -\frac{1}{2a^2}h_{00,ii}^{(1,2)} + \frac{1}{2a^2}h_{00,ij}^{(0,2)}h_{ij}^{(1,0)} \\ & + \text{ terms of size } \left[\frac{\epsilon\eta^2}{L_N L_C}\right] \\ \sim & \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_N L_C}\,. \end{aligned} \tag{A.10}$$

The terms in Eq. (A.2) are given by

$$R^{(0,2)}_{0i} = -\frac{\dot{a}}{a} h^{(0,2)}_{00,i} \sim \frac{\eta^2}{L_C L_N} \tag{A.11}$$

$$\begin{aligned} R^{(0,3)}_{0i} &= \frac{1}{2a}\left(h^{(0,3)}_{0j,ij} - h^{(0,3)}_{0i,jj} + a h^{(0,2)}_{ij,0j} - a h^{(0,2)}_{jj,0i}\right) \\ &\qquad + \text{ terms of size } \left[\frac{\epsilon\eta^3}{L_C^2}\right] \\ &\sim \frac{\eta^3}{L_N^2} + \frac{\eta^3}{L_C^2} \end{aligned} \tag{A.12}$$

$$\begin{aligned} R^{(1,0)}_{0i} &= \frac{1}{2a}\Big(h^{(1,0)}_{0j,ij} - h^{(1,0)}_{0i,jj} + a h^{(1,0)}_{ij,0j} - a h^{(1,0)}_{jj,0i} \\ &\quad - 2\dot{a} h^{(1,0)}_{00,i} + 4\dot{a}^2 h^{(1,0)}_{0i} + 2a\ddot{a} h^{(1,0)}_{0i}\Big) \sim \frac{\epsilon}{L_C^2} \end{aligned} \tag{A.13}$$

$$R^{(1,1)}_{0i} = -2\dot{a} h^{(1,1)}_{00,i} \sim \frac{\epsilon\eta}{L_N L_C} \tag{A.14}$$

$$\begin{aligned} R^{(1,2)}_{0i} &= \frac{1}{2a}\left(h^{(1,2)}_{0j,ij} - h^{(1,2)}_{0i,jj} + a h^{(1,1)}_{ij,0j} - a h^{(1,1)}_{jj,0i}\right) \\ &\quad + \frac{1}{2a} h^{(1,0)}_{0j} h^{(0,2)}_{00,ij} \\ &\qquad + \text{ terms of size } \left[\frac{\epsilon\eta^2}{L_C^2} + \frac{\epsilon\eta^2}{L_N L_C}\right] \\ &\sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2} + \frac{\epsilon\eta^2}{L_N L_C}\,. \end{aligned} \tag{A.15}$$

Finally, the terms in Eq. (A.3) are given by

$$R_{ij}^{(0,0)} = \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{ij} \sim \frac{1}{L_C^2}, \tag{A.16}$$

$$\begin{aligned} R_{ij}^{(0,2)} &= \frac{1}{2}\left(h_{00,ij}^{(0,2)} + 2h_{k(i,j)k}^{(0,2)} - h_{kk,ij}^{(0,2)} - h_{ij,kk}^{(0,2)}\right) \\ &+ \left(2\dot{a}^2 + a\ddot{a}\right)\left(h_{ij}^{(0,2)} + h_{00}^{(0,2)}\delta_{ij}\right) \\ &\sim \frac{\eta^2}{L_N^2} + \frac{\eta^2}{L_C^2} \end{aligned} \tag{A.17}$$

$$\begin{aligned} R_{ij}^{(0,3)} &= \frac{1}{2}a\dot{a}h_{00,0}^{(0,2)}\delta_{ij} - 2\dot{a}h_{0(i,j)}^{(0,3)} - \dot{a}h_{0k,k}^{(0,3)}\delta_{ij} \\ &+ \frac{3}{2}a\dot{a}h_{ij,0}^{(0,2)} + \frac{1}{2}a\dot{a}h_{kk,0}^{(0,2)}\delta_{ij} \sim \frac{\eta^3}{L_C L_N} \end{aligned} \tag{A.18}$$

$$\begin{aligned} R_{ij}^{(0,4)} &= \frac{1}{2}\left(h_{00,ij}^{(0,4)} - h_{ij,kk}^{(0,4)} - h_{kk,ij}^{(0,4)}\right) + h_{k(i,j)k}^{(0,4)} \\ &+ a^2 h_{ij,00}^{(0,2)} + \frac{1}{2}h_{00,k}^{(0,2)}\left(h_{ij,k}^{(0,2)} - 2h_{k(i,j)}^{(0,2)}\right) \\ &\quad + h_{kl,ij}^{(0,2)}h_{kl}^{(0,2)} + h_{ij,kl}^{(0,2)}h_{kl}^{(0,2)} - 2h_{k(i,j)l}^{(0,2)}h_{kl}^{(0,2)} \\ &\quad + \frac{1}{2}h_{kl,i}^{(0,2)}h_{kl,j}^{(0,2)} + h_{kl,l}^{(0,2)}\left(h_{ij,k}^{(0,2)} - 2h_{k(i,j)}^{(0,2)}\right) \\ &\quad + h_{ik,l}^{(0,2)}\left(h_{jk,l}^{(0,2)} - h_{jl,k}^{(0,2)}\right) + \frac{1}{2}h_{00,i}^{(0,2)}h_{00,j}^{(0,2)} \\ &+ h_{00,ij}^{(0,2)}h_{00}^{(0,2)} + h_{kk,l}^{(0,2)}\left(2h_{l(i,j)}^{(0,2)} - h_{ij,l}^{(0,2)}\right) \\ &- 2ah_{0(i,j)0}^{(0,3)} + \text{terms of size}\left[\frac{\eta^4}{L_C^2}\right] \\ &\sim \frac{\eta^4}{L_N^2} + \frac{\eta^4}{L_C^2} \end{aligned} \tag{A.19}$$

$$\begin{aligned} R_{ij}^{(1,0)} &= \frac{1}{2}\left(h_{00,ij}^{(1,0)} - h_{ij,kk}^{(1,0)} - h_{kk,ij}^{(1,0)}\right) + h_{k(i,j)k}^{(1,0)} \\ &+ a\ddot{a}h_{ij}^{(1,0)} + a\ddot{a}h_{00}^{(1,0)}\delta_{ij} + 2\dot{a}^2 h_{00}^{(1,0)}\delta_{ij} \\ &+ \frac{1}{2}a\dot{a}h_{00,0}^{(1,0)}\delta_{ij} - 2\dot{a}h_{0(i,j)}^{(1,0)} - \dot{a}h_{0k,k}^{(1,0)}\delta_{ij} \\ &+ \frac{3}{2}a\dot{a}h_{ij,0}^{(1,0)} + \frac{1}{2}a\dot{a}h_{kk,0}^{(1,0)}\delta_{ij} + \frac{1}{2}a^2 h_{ij,00}^{(1,0)} \\ &+ 2\dot{a}^2 h_{ij}^{(1,0)} - ah_{0(i,j)0}^{(1,0)} \sim \frac{\epsilon}{L_C^2} \end{aligned} \tag{A.20}$$

(A.21)

$$\begin{aligned}
R_{ij}^{(1,1)} &= \frac{1}{2}(h_{00,ij}^{(1,1)} - h_{ij,kk}^{(1,1)} - h_{kk,ij}^{(1,1)}) + h_{k(i,j)k}^{(1,1)} \\
&\quad + \text{ terms of size } \left[\frac{\epsilon\eta}{L_C^2}\right] \\
&\sim \frac{\epsilon\eta}{L_N^2} + \frac{\epsilon\eta}{L_C^2}
\end{aligned} \tag{A.22}$$

$$\begin{aligned}
R_{ij}^{(1,2)} &= \frac{1}{2}\left(h_{00,ij}^{(1,2)} - h_{ij,kk}^{(1,2)} - h_{kk,ij}^{(1,2)}\right) + h_{k(i,j)k}^{(1,2)} \\
&\quad + \frac{1}{2}h_{00,ij}^{(0,2)}h_{00}^{(1,0)} + \frac{1}{2}h_{kl,ij}^{(0,2)}h_{kl}^{(1,0)} + \frac{1}{2}h_{ij,kl}^{(0,2)}h_{kl}^{(1,0)} \\
&\quad - h_{k(i,j)l}^{(0,2)}h_{kl}^{(1,0)} \\
&\quad + \text{ terms of size } \left[\frac{\epsilon\eta^2}{L_C^2} + \frac{\epsilon\eta^2}{L_N L_C}\right] \\
&\sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2} + \frac{\epsilon\eta^2}{L_N L_C},
\end{aligned} \tag{A.23}$$

where in Eq. (A.17) the two orders or magnitude after the $\sim$ indicate the sizes of the first and second lines, respectively.

We now perform the same exercise for the stress-energy tensor:

$$T_{00} = T_{00}^{(0,0)} + T_{00}^{(0,2)} + \tfrac{1}{2}T_{00}^{(0,4)} + T_{00}^{(1,0)} + T_{00}^{(1,1)} + T_{00}^{(1,2)} + \dots \tag{A.24}$$

$$T_{ij} = T_{ij}^{(0,0)} + T_{ij}^{(0,2)} + T_{ij}^{(1,0)} + T_{ij}^{(1,1)} + T_{ij}^{(1,2)} + \tfrac{1}{2}T_{ij}^{(0,4)} \dots \tag{A.25}$$

$$T_{0i} = T_{0i}^{(0,1)} + T_{0i}^{(0,3)} + T_{0i}^{(1,0)} + T_{0i}^{(1,2)} + \dots. \tag{A.26}$$

The terms on the right-hand side of Eq. (A.24) are given by

$$T_{00}^{(0,0)} = \rho^{(0,0)} \sim \frac{1}{L_C^2} \tag{A.27}$$

$$T_{00}^{(0,2)} = \rho^{(0,2)} - \rho^{(0,0)} h_{00}^{(0,2)} + (\rho^{(0,0)} + P^{(0,0)}) v^{(0,1)i} v_i^{(0,1)} \sim \frac{\eta^2}{L_N^2} + \frac{\eta^2}{L_C^2} \tag{A.28}$$

$$\begin{aligned} T_{00}^{(0,4)} &= \frac{1}{2}\rho^{(0,4)} - h_{00}^{(0,2)}\rho^{(0,2)} + \rho^{(0,2)} v^{(0,1)i} v_i^{(0,1)} \\ &\quad + \text{terms of size } \left[\frac{\eta^4}{L_C^2}\right] \sim \frac{\eta^4}{L_N^2} + \frac{\eta^4}{L_C^2} \end{aligned} \tag{A.29}$$

$$T_{00}^{(1,0)} = \rho^{(1,0)} - \rho^{(0,0)} h_{00}^{(1,0)} \sim \frac{\epsilon}{L_C^2} \tag{A.30}$$

$$T_{00}^{(1,1)} = \rho^{(1,1)} + \text{terms of size } \left[\frac{\epsilon\eta}{L_C^2}\right] \sim \frac{\epsilon\eta}{L_N^2} + \frac{\epsilon\eta}{L_C^2} \tag{A.31}$$

$$T_{00}^{(1,2)} = \rho^{(1,2)} - h_{00}^{(1,0)}\rho^{(0,2)} + \text{terms of size } \left[\frac{\epsilon\eta^2}{L_C^2}\right] \sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2}\,, \tag{A.32}$$

while the terms in Eq. (A.25) are given by

$$T_{ij}^{(0,0)} = a^2 P^{(0,0)} \delta_{ij} \sim \frac{1}{L_C^2} \tag{A.33}$$

$$T_{ij}^{(0,2)} = a^2 \left(\rho^{(0,0)} + P^{(0,0)}\right) v_i^{(0,1)} v_j^{(0,1)} + a^2 P^{(0,0)} h_{ij}^{(0,2)} \sim \frac{\eta^2}{L_C^2} \tag{A.34}$$

$$T_{ij}^{(1,0)} = a^2 P^{(1,0)} \delta_{ij} + a^2 h_{ij}^{(1,0)} P^{(0,0)} \sim \frac{\epsilon}{L_C^2} \tag{A.35}$$

$$T_{ij}^{(1,1)} = \text{terms of size } \left[\frac{\epsilon\eta}{L_C^2}\right] \tag{A.36}$$

$$T_{ij}^{(1,2)} = a^2 P^{(1,2)} \delta_{ij} + \text{terms of size } \left[\frac{\epsilon\eta^2}{L_C^2}\right] \sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2} \tag{A.37}$$

$$T_{ij}^{(0,4)} = a^2 \rho^{(0,2)} v_i^{(0,1)} v_j^{(0,1)} + \frac{a^2}{2} P^{(0,4)} \delta_{ij} + \text{terms of size } \left[\frac{\eta^4}{L_C^2}\right] \sim \frac{\eta^4}{L_N^2} + \frac{\eta^4}{L_C^2}\,, \tag{A.38}$$

and the terms in Eq. (A.26) are given by

$$T_{0i}^{(0,1)} = -a(\rho^{(0,0)} + P^{(0,0)})v_i^{(0,1)} \sim \frac{\eta}{L_C^2} \tag{A.39}$$

$$T_{0i}^{(0,3)} = -a\rho^{(0,2)}v_i^{(0,1)} + \text{ terms of size } \left[\frac{\eta^3}{L_C^2}\right] \sim \frac{\eta^3}{L_N^2} + \frac{\eta^3}{L_C^2} \tag{A.40}$$

$$T_{0i}^{(1,0)} = -a\rho^{(0,0)}(v_i^{(1,0)} + h_{0i}^{(1,0)}) - aP^{(0,0)}v_i^{(1,0)} \sim \frac{\epsilon}{L_C^2} \tag{A.41}$$

$$\begin{aligned} T_{0i}^{(1,2)} &= -a\rho^{(0,2)}\left(v_i^{(1,0)} + h_{0i}^{(1,0)}\right) - a\rho^{(1,1)}v_i^{(0,1)} \\ &\quad + \text{ terms of size } \left[\frac{\epsilon\eta^2}{L_C^2}\right] \sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2}\,. \end{aligned} \tag{A.42}$$

We have now provided all the expressions for the perturbed Ricci tensor and stress-energy tensor components required to calculate the field equations.

# B. Two-parameter gauge invariant metric potentials

In this appendix, we repeat the derivation of the two-parameter gauge invariant metric potentials first presented in [125]. We stress that this material is repeated here so that a fully self-consistent presentation of the two-parameter perturbation theory is available to the reader, and it does not comprise an original part of the work presented in this thesis.

## B.1. Infinitesimal coordinate transformations

To perform a gauge transformation, it is useful to split the perturbed gauge generator into a scalar component and a divergenceless vector component. Without superscripts, these are denoted

$$\xi^0 \equiv \delta t \qquad \text{and} \qquad \xi^i \equiv \delta x_{,}{}^{i} + \delta x^i \,, \tag{B.1}$$

where $\delta x^i{}_{,i} = 0$. In the remainder of this section we will outline how the presence of radiation affects the transformation properties of the matter fields $\{\rho, P, v_i, \Lambda\}$. This is done using the result from Eq. (6.26), and by solving for the decomposed matter variables.

In order to present these results in a form that can be used for cosmology we choose to take $L_N/L_C \sim \eta$. This means that we are restricting the post-Newtonian sector of our expansion to apply on scales below about 100Mpc, which is realistically also about the size of the homogeneity scale. This is ideal for considering the influence of galaxies, clusters and super-clusters on large-scale linear cosmological perturbations. We also choose, without loss of generality, to express our results in terms of $L_N$.

Given this, the total energy density transforms as follows:

$$\tilde{\rho}^{(0,0)} + \tilde{\rho}^{(0,2)} = \rho^{(0,0)} + \rho^{(0,2)} \sim \frac{\eta^2}{L_N^2} \tag{B.2}$$

$$\tilde{\rho}^{(1,1)} = \rho^{(1,1)} + \rho^{(0,2)}_{,i}\left(\delta x^{(1,0)\,i}_{\quad\;\;,} + \delta x^{(1,0)i}\right) \sim \frac{\epsilon\eta}{L_N^2} \tag{B.3}$$

$$\tilde{\rho}^{(1,0)} + \tilde{\rho}^{(1,2)} = \rho^{(1,0)} + \rho^{(1,2)} + \left(\rho^{(0,0)} + \rho^{(0,2)}\right)^{\cdot} \delta t^{(1,0)} \sim \frac{\epsilon\eta^2}{L_N^2} \tag{B.4}$$

$$\tilde{\rho}^{(0,4)} = \rho^{(0,4)} + 2\rho^{(0,2)}_{,i}\left(\delta x^{(0,2)\,i}_{\quad\;\;,} + \delta x^{(0,2)i}\right) \sim \frac{\eta^4}{L_N^2}\,, \tag{B.5}$$

while the total pressure transforms as

$$\tilde{p}^{(0,0)} = P^{(0,0)} \sim \frac{\eta^2}{L_N^2} \tag{B.6}$$

$$\tilde{p}^{(1,0)} + \tilde{p}^{(1,2)} = P^{(1,0)} + P^{(1,2)} + \dot{P}^{(0,0)}\delta t^{(1,0)0} - 2\frac{\dot{a}}{a}P^{(0,0)}\delta t^{(1,0)0} \sim \frac{\epsilon\eta^2}{L_N^2} \tag{B.7}$$

$$\tilde{p}^{(0,4)} = P^{(0,4)} \sim \frac{\eta^4}{L_N^2}\,. \tag{B.8}$$

The transformations in Eqs. (B.3), (B.5) and (B.8) remain exactly the same as the dust-only case studied in Ref. [125], while all other transformations are affected by the presence of the radiation. The term $\rho^{(0,0)}$ can be seen to transform in the same was as the Newtonian energy density, $\rho^{(0,2)}$. This is not unexpected, as both quantities have magnitude $\sim L_C^{-2} \sim \eta^2 L_N^{-2}$. Similarly, $\rho^{(0,0)}$ appears alongside $\rho^{(0,2)}$ in the transformation given in Eq. (B.4). With the inclusion of radiation, we find that $P^{(0,0)}$ is automatically gauge invariant. Furthermore, as can be seen in Eq. (B.7), the inclusion of radiation means that the transformation of the cosmological and mixed-order perturbations to the pressure are no longer gauge invariant (as they were in the dust-only case). The reader may note that these results differ from the quasi-static limit of cosmological perturbation theory, as space and time derivatives are treated on a different footing, and because velocities come in at different orders [77].

Meanwhile, the peculiar velocities transform in the following way:

$$\tilde{v}_i^{(1,0)} = v_i^{(1,0)} - a\left(\delta x^{(1,0)}_{,i} + \delta x^{(1,0)}_i\right)^{\cdot} + v^{(0,1)}_{i,j}\left(\delta x^{(1,0)\,j}_{\quad\;\;,} + \delta x^{(1,0)j}\right) \sim \epsilon \tag{B.9}$$

$$\tilde{v}_i^{(0,1)} = v_i^{(0,1)} \sim \eta\,. \tag{B.10}$$

These transformations are the same as in the dust-only case studied in Ref. [125]. Note particularly that in Eq. (B.9) the small-scale Newtonian velocity contributes to the transformation of the large-scale velocity – this is a by-product of our two-parameter expansion, and is an effect that would otherwise only appear at second order in standard cosmological perturbation theory. Finally, we find that the cosmological constant $\Lambda^{(0,0)}$ does not transform under the gauge transformation in Eq. (6.25), as it is a constant in space and time:

$$\tilde{\Lambda}^{(0,0)} = \Lambda^{(0,0)} \, . \tag{B.11}$$

The transformations above will now be used to construct gauge-invariant quantities.

## B.2. Gauge invariant matter variables

Let us now create gauge-invariant quantities for the matter degrees of freedom in the presence of radiation and $\Lambda$. Such variables isolate and remove superfluous degrees of freedom, as well as allowing the field equations to be written in a greatly simplified way. To do this it is useful to perform an irreducible decomposition on the metric. Omitting superscripts for simplicity, and without loss of generality, we can do this as follows:

$$h_{00} \equiv \phi \, , \qquad h_{0i} \equiv B_{,i} + B_i \qquad \text{and} \qquad h_{ij} \equiv -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \tfrac{1}{2}\hat{h}_{ij} \, , \tag{B.12}$$

where $B_i \, , F_i \, , \hat{h}_{ij}$ are divergenceless and $\hat{h}_{ij}$ is trace-free. Applying the gauge transformation (6.26) to the metric components (6.6)-(6.8) this gives the transformation rules for the irreducibly decomposed components, and allows gauge-invariant gravitational perturbations to be constructed (see Sections V and VI of Ref. [125]). The presence of radiation does not affect the construction of gauge-invariant gravitational perturbations, but does affect the construction of gauge-invariant quantities for the matter variables, which is what we will elaborate upon here.

The method we use to calculate gauge-invariant quantities is as follows: we choose gauge generators $\delta x, \delta x^i$ and $\delta t$ such that the gauge transformed metric potentials $\tilde{E} = \tilde{B} = \tilde{F}_i = 0$. We then substitute these gauge generators, now written in terms of $E, B$ and $F_i$, back into the expressions for all of the transformed perturbations presented in Section 6.3. Because the original gauge transformations were written down in a completely arbitrary coordinate system, these new results are automatically gauge invariant [31]. All such quantities also reduce to metric perturbations in longitudinal gauge when $E = B = F_i = 0$, and have been explicitly checked to be truly gauge invariant.

To construct gauge-invariant matter perturbations we require the transformation laws for $E^{(1,0)}, B^{(1,0)}, F^{(1,0)i}, E^{(0,2)}$ and $F^{(0,2)i}$ under Eq. (6.25). These are given in [125] and are

$$\tilde{B}^{(1,0)} = B^{(1,0)} + a\dot{\delta x}^{(1,0)} - \tfrac{1}{a}\delta t^{(1,0)} \sim \epsilon\eta^{-1}L_N \tag{B.13}$$
$$\tilde{E}^{(1,0)} = E^{(1,0)} + 2\delta x^{(1,0)} \sim \epsilon\eta^{-2}L_N^2 \tag{B.14}$$
$$\tilde{F}_i^{(1,0)} = F_i^{(1,0)} + 2\delta x_i^{(1,0)} \sim \epsilon\eta^{-1}L_N \tag{B.15}$$
$$\tilde{E}^{(0,2)} = E^{(0,2)} + 2\delta x^{(0,2)} \sim \eta^2 L_N^2 \tag{B.16}$$
$$\tilde{F}_i^{(0,2)} = F_i^{(0,2)} + 2\delta x_i^{(0,2)} \sim \eta^2 L_N\,. \tag{B.17}$$

For the total energy density perturbations it can then be seen that the following quantities are gauge invariant:

$$\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} = \rho^{(0,0)} + \rho^{(0,2)} \tag{B.18}$$
$$\boldsymbol{\rho}^{(1,1)} = \rho^{(1,1)} - \frac{1}{2}\rho^{(0,2)}_{,i}\left(E^{(1,0),i} + F^{(1,0)i}\right) \tag{B.19}$$
$$\boldsymbol{\rho}^{(1,0)} + \boldsymbol{\rho}^{(1,2)} = \rho^{(1,0)} + \rho^{(1,2)} + \left(\rho^{(0,0)} + \rho^{(0,2)}\right)^{\cdot}\left(aB^{(1,0)} - \frac{a^2}{2}\dot{E}^{(1,0)}\right) \tag{B.20}$$
$$\boldsymbol{\rho}^{(0,4)} = \rho^{(0,4)} - \rho^{(0,2)}_{,i}\left(E^{(0,2),i} + F^{(0,2)i}\right)\,. \tag{B.21}$$

Correspondingly, for the pressure perturbations we find the following gauge-invariant quantities:

$$\mathbf{P}^{(0,0)} = P^{(0,0)} \tag{B.22}$$
$$\mathbf{P}^{(1,0)} + \mathbf{p}^{(1,2)} = P^{(1,0)} + P^{(1,2)} + \left(\dot{p}^{(0,0)} - 2\frac{\dot{a}}{a}P^{(0,0)}\right)\left(aB^{(1,0)} - \frac{a^2}{2}\dot{E}^{(1,0)}\right) \tag{B.23}$$
$$\mathbf{P}^{(0,4)} = P^{(0,4)}\,, \tag{B.24}$$

and for the peculiar velocity we construct

$$\mathbf{v}_i^{(0,1)} = v_i^{(0,1)} \tag{B.25}$$
$$\mathbf{v}_i^{(1,0)} = v_i^{(1,0)} + \frac{a}{2}\left(\dot{E}_{,i}^{(1,0)} + \dot{F}_i^{(1,0)}\right) - \frac{1}{2}v_{i,j}^{(0,1)}\left(E^{(1,0),j} + F^{(1,0)j}\right)\,. \tag{B.26}$$

These last two quantities can be separated into scalar and divergenceless vector parts in a straightforward way. Finally, the gauge-invariant cosmological constant is trivial to construct:

$$\mathbf{\Lambda} = \Lambda^{(0,0)} \,. \tag{B.27}$$

There are no further quantities to consider in the stress-energy tensor, so, when combined with the set of gauge-invariant metric potentials, constructed in Section B.3, this yields a full set of gauge-invariant quantities in the two-parameter perturbative expansion.

## B.3. Transformation of metric components

**The time-time component:** The fluctuations in the time-time component of the metric, up to the required order, transform under the exponential map in Eq. (6.26) as:

$$h_{00}^{(0,2)} \mapsto \tilde{h}_{00}^{(0,2)} = h_{00}^{(0,2)} \tag{B.28}$$

$$h_{00}^{(1,0)} \mapsto \tilde{h}_{00}^{(1,0)} = h_{00}^{(1,0)} - 2\dot{\xi}^{(1,0)0} \tag{B.29}$$

$$h_{00}^{(1,1)} \mapsto \tilde{h}_{00}^{(1,1)} = h_{00}^{(1,1)} + h_{00,i}^{(0,2)}\xi^{(1,0)i} \tag{B.30}$$

$$h_{00}^{(1,2)} \mapsto \tilde{h}_{00}^{(1,2)} = h_{00}^{(1,2)} + \dot{h}_{00}^{(0,2)}\xi^{(1,0)0} + 2h_{00}^{(0,2)}\dot{\xi}^{(1,0)0} \tag{B.31}$$

$$h_{00}^{(0,4)} \mapsto \tilde{h}_{00}^{(0,4)} = h_{00}^{(0,4)} - 4\dot{\xi}^{(0,3)0} + 2h_{00,i}^{(0,2)}\xi^{(0,2)i} \,. \tag{B.32}$$

We note that there is also a term generated from Eq. (6.26) in this component of the metric that is

$$\tfrac{1}{2}h_{00,ij}^{(0,2)}\xi^{(1,0)i}\xi^{(1,0)j} \,, \tag{B.33}$$

which is of the $\mathcal{O}(\epsilon^2)$ when the length scales are taken into account appropriately. However, this term appears in the $\mathcal{O}(\eta^4 L_N^{-2})$ 00-field equation, Eq. (C.7), in the form of $R_{\mu\nu}^{(2,0)} \sim \frac{1}{2}\nabla^2(h_{00,ij}^{(0,2)}\xi^{(1,0)i}\xi^{(1,0)j}) \sim \frac{1}{2}\nabla^2 h_{00,ij}^{(0,2)}\xi^{(1,0)i}\xi^{(1,0)j} \sim \epsilon^2 L_N^{-2} \sim \eta^4 L_N^{-2}$, when $\epsilon \sim \eta^2$. This term however cancels with an analogous term generated for $T_{\mu\nu}^{(2,0)}$ in the field equation, and does not contribute any new dynamics.

**The time-space components:** The perturbations of the time-space parts of the metric transform, according to Eq. (6.26), in the following way:

$$h_{0i}^{(0,3)} \mapsto \tilde{h}_{0i}^{(0,3)} = h_{0i}^{(0,3)} - \frac{1}{a}\xi_{,i}^{(0,3)0} + a\dot{\xi}_i^{(0,2)} \tag{B.34}$$

$$h_{0i}^{(1,0)} \mapsto \tilde{h}_{0i}^{(1,0)} = h_{0i}^{(1,0)} - \frac{1}{a}\xi_{,i}^{(1,0)0} + a\dot{\xi}_i^{(1,0)} \tag{B.35}$$

$$h_{0i}^{(1,2)} \mapsto \tilde{h}_{0i}^{(1,2)} = h_{0i}^{(1,2)} - \frac{1}{a}\xi_{,i}^{(1,2)0} + a\dot{\xi}_i^{(1,1)} + \chi_i^{(1,2)}\,, \tag{B.36}$$

where we define

$$\begin{aligned}\chi_i^{(1,2)} \equiv\ & \frac{1}{a}h_{00}^{(0,2)}\xi_{,i}^{(1,0)0} + a\left(h_{ij}^{(0,2)} + \xi_{(i,j)}^{(0,2)}\right)\dot{\xi}^{(1,0)j} + \left(h_{0i}^{(0,3)} - \frac{1}{2a}\xi_{,i}^{(0,3)0} + \frac{1}{2}a\dot{\xi}_i^{(0,2)}\right)_{,j}\xi^{(1,0)j} \\ & + \left(h_{0j}^{(1,0)} - \frac{1}{2a}\xi_{,j}^{(1,0)0} + \frac{1}{2}a\dot{\xi}_j^{(1,0)}\right)\xi_{,i}^{(0,2)j}\,. \end{aligned} \tag{B.37}$$

**The space-space components:** The transformations of the perturbations in the space-space part of the metric are more lengthy than the previous cases. They transform under the exponential map in Eq. (6.26) in the following way:

$$h_{ij}^{(0,2)} \mapsto \tilde{h}_{ij}^{(0,2)} = h_{ij}^{(0,2)} + 2\xi_{(i,j)}^{(0,2)} \tag{B.38}$$

$$h_{ij}^{(1,0)} \mapsto \tilde{h}_{ij}^{(1,0)} = h_{ij}^{(1,0)} + 2\frac{\dot{a}}{a}\xi^{(1,0)0}\delta_{ij} + 2\xi_{(i,j)}^{(1,0)} \tag{B.39}$$

$$h_{ij}^{(1,1)} \mapsto \tilde{h}_{ij}^{(1,1)} = h_{ij}^{(1,1)} + 2\xi_{(i,j)}^{(1,1)} + \chi_{ij}^{(1,1)} \tag{B.40}$$

$$h_{ij}^{(1,2)} \mapsto \tilde{h}_{ij}^{(1,2)} = h_{ij}^{(1,2)} + 2\xi_{(i,j)}^{(1,2)} + \chi_{ij}^{(1,2)} \tag{B.41}$$

$$h_{ij}^{(0,4)} \mapsto \tilde{h}_{ij}^{(0,4)} = h_{ij}^{(0,4)} + 4\frac{\dot{a}}{a}\xi^{(0,3)0}\delta_{ij} + 2\xi_{(i,j)}^{(0,4)} + \chi_{ij}^{(0,4)}\,, \tag{B.42}$$

where $\chi_{ij}^{(1,1)}$, $\chi_{ij}^{(1,2)}$ and $\chi_{ij}^{(0,4)}$ are defined as

$$\chi_{ij}^{(1,1)} \equiv \left(h_{ij}^{(0,2)} + 2\xi_{(i,j)}^{(0,2)}\right)_{,k}\xi^{(1,0)k} \tag{B.43}$$

$$\begin{aligned}\chi_{ij}^{(1,2)} \equiv\ & \left(h_{ij}^{(0,2)} + \xi_{(i,j)}^{(0,2)}\right)\dot{\text{ﬄ}}\,\xi^{(1,0)0} + 2\frac{\dot{a}}{a}\left(h_{ij}^{(0,2)} + 2\xi_{(i,j)}^{(0,2)}\right)\xi^{(1,0)0} + \left(h_{ik}^{(0,2)} + \xi_{(i,k)}^{(0,2)}\right)\xi_{,j}^{(1,0)k} \\ & + \left(h_{jk}^{(0,2)} + \xi_{(j,k)}^{(0,2)}\right)\xi_{,i}^{(1,0)k} + \left(h_{ik}^{(1,0)} + \xi_{(i,k)}^{(1,0)}\right)\xi_{,j}^{(0,2)k} + \left(h_{jk}^{(1,0)} + \xi_{(j,k)}^{(1,0)}\right)\xi_{,i}^{(0,2)k} \end{aligned} \tag{B.44}$$

$$\chi_{ij}^{(0,4)} \equiv 2\left(h_{ij}^{(0,2)} + \xi_{(i,j)}^{(0,2)}\right)_{,k}\xi^{(0,2)k} + 2\left(h_{ik}^{(0,2)} + \xi_{(i,k)}^{(0,2)}\right)\xi_{,j}^{(0,2)k} + 2\left(h_{jk}^{(0,2)} + \xi_{(j,k)}^{(0,2)}\right)\xi_{,i}^{(0,2)k}\,. \tag{B.45}$$

Before finishing this section, let us comment on the dependence of some of these terms on the condition $L_N/L_C \sim \eta$. In the time-time transformation the only terms that depend on this relation are $h^{(0,2)}_{00,i}\xi^{(1,0)i}$ and $\dot{h}^{(0,2)}_{00}\xi^{(1,0)0}$, which, once length scales are taken into account properly, appear at $\mathcal{O}(\epsilon\eta)$ and $\mathcal{O}(\epsilon\eta^2)$, respectively. If a different relationship between $L_N$ and $L_C$ had been chosen then this term would have appeared at a different order, and could appear in any equation greater than or equal to $\epsilon\eta$ and $\epsilon\eta^2$, respectively. Similarly, in the transformation of the time-space and space-space components of the metric some of the terms in $\chi^{(1,2)}_i$ and $\chi^{(1,2)}_{ij}$, and terms $4\frac{\dot{a}}{a}\xi^{(0,3)0}\delta_{ij}$ and $\chi^{(1,1)}_{ij}$, all depend on the relationship between $L_N$ and $L_C$, and would appear at different orders if a different choice had been made for these length scales.

## B.4. Transformation of irreducibly-decomposed potentials

Having performed the gauge transformation of our metric components, in the previous section, we can now perform an irreducible decomposition of these objects into scalars, divergenceless vectors ($V^i{}_{,i} = 0$), and transverse and trace-free tensors ($h^i{}_i = 0$ and $h^{ij}{}_{,j} = 0$). These are the quantities that are most often considered in cosmological perturbation theory, and that usually decouple from each at first order in perturbations. We decompose our metric potentials into these variables in the following way, omitting superscripts for simplicity:

$$h_{00} \equiv \phi\,, \qquad h_{0i} \equiv B_{,i} + B_i \qquad \text{and} \qquad h_{ij} \equiv -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \tfrac{1}{2}\hat{h}_{ij}\,. \tag{B.46}$$

Similarly, our gauge generators will be decomposed such that

$$\xi^0 \equiv \delta t \qquad \text{and} \qquad \xi^i \equiv \delta x^{,i} + \delta x^i\,. \tag{B.47}$$

We will now present the result of gauge transformations on each of the irreducibly decomposed objects, in each of the sectors of our perturbation theory.

**Cosmological scalar, vector and tensor potentials:** The gauge transformations given in Eqs. (B.29), (B.35), and (B.39) now allow us to write down the transformation of the decomposed metric components in the cosmological sector of our theory. For the scalar potentials these transformations are given by

$$\tilde{\phi}^{(1,0)} = \phi^{(1,0)} - 2\dot{\delta t}^{(1,0)} \sim \epsilon \tag{B.48}$$

$$\tilde{\psi}^{(1,0)} = \psi^{(1,0)} - 2\frac{\dot{a}}{a}\delta t^{(1,0)} \sim \epsilon \tag{B.49}$$

$$\tilde{B}^{(1,0)} = B^{(1,0)} + a\dot{\delta x}^{(1,0)} - \frac{1}{a}\delta t^{(1,0)} \sim \epsilon\eta^{-1}L_N \tag{B.50}$$

$$\tilde{E}^{(1,0)} = E^{(1,0)} + 2\delta x^{(1,0)} \sim \epsilon\eta^{-2}L_N^2\,, \tag{B.51}$$

for the vector potentials they are

$$\tilde{B}_i^{(1,0)} = B_i^{(1,0)} + a\dot{\delta x_i}^{(1,0)} \sim \epsilon \tag{B.52}$$

$$\tilde{F}_i^{(1,0)} = F_i^{(1,0)} + 2\delta x_i^{(1,0)} \sim \epsilon\eta^{-1}L_N\,, \tag{B.53}$$

and for the tensor potential this transformation is

$$\tilde{\hat{h}}_{ij}^{(1,0)} = \hat{h}_{ij}^{(1,0)} \sim \epsilon\,. \tag{B.54}$$

As in previous equations, the quantity after the $\sim$ sign gives the order of each of these potentials in terms of $\epsilon$, $\eta$ and any relevant length scales. We observe that the transformation of the above cosmological scalar, vector and tensor potentials in our two-parameter formalism are the same as those derived from linear cosmological perturbation theory [31], perturbed in one parameter.

**Post-Newtonian scalar, vector and tensor potentials:** The results given in Eqs. (B.28), (B.32), (B.34), (B.38), and (B.42) allow us to write the transformation of the decomposed post-Newtonian potentials. The scalar parts of the post-

Newtonian potentials transform as

$$\tilde{\phi}^{(0,2)} = \phi^{(0,2)} \sim \eta^2 \tag{B.55}$$

$$\tilde{\phi}^{(0,4)} = \phi^{(0,4)} - 4\dot{\delta t}^{(0,3)} + 2\phi^{(0,2)}_{,i}\left(\delta x^{(0,2),i} + \delta x^{(0,2)i}\right) \sim \eta^4 \tag{B.56}$$

$$\tilde{\psi}^{(0,2)} = \psi^{(0,2)} \sim \eta^2 \tag{B.57}$$

$$\tilde{\psi}^{(0,4)} = \psi^{(0,4)} - 4\frac{\dot{a}}{a}\delta t^{(0,3)} + \frac{1}{2}\left(\nabla^{-2}\chi^{(0,4),ij}_{ij} - \chi^{(0,4)}\right) \sim \eta^4 \tag{B.58}$$

$$\tilde{B}^{(0,3)} = B^{(0,3)} + a\dot{\delta x}^{(0,2)} - \frac{1}{a}\delta t^{(0,3)} \sim \eta^3 L_N \tag{B.59}$$

$$\tilde{E}^{(0,2)} = E^{(0,2)} + 2\delta x^{(0,2)} \sim \eta^2 L_N^2 \tag{B.60}$$

$$\tilde{E}^{(0,4)} = E^{(0,4)} + 2\delta x^{(0,4)} + \frac{1}{2}\nabla^{-2}\left(3\nabla^{-2}\chi^{(0,4),ij}_{ij} - \chi^{(0,4)}\right) \sim \eta^4 L_N^2\,, \tag{B.61}$$

the vector potentials transform as

$$\tilde{B}^{(0,3)}_i = B^{(0,3)}_i + a\dot{\delta x_i}^{(0,2)} \sim \eta^3 \tag{B.62}$$

$$\tilde{F}^{(0,2)}_i = F^{(0,2)}_i + 2\delta x^{(0,2)}_i \sim \eta^2 L_N \tag{B.63}$$

$$\tilde{F}^{(0,4)}_i = F^{(0,4)}_i + 2\delta x^{(0,4)}_i + 2\nabla^{-2}\left(\chi^{(0,4),k}_{ik} - \nabla^{-2}\chi^{(0,4),kj}_{kj,i}\right) \sim \eta^4 L_N\,, \tag{B.64}$$

and the tensor potentials transform as

$$\tilde{\hat{h}}^{(0,2)}_{ij} = \hat{h}^{(0,2)}_{ij} \sim \eta^2 \tag{B.65}$$

$$\begin{aligned}\tilde{\hat{h}}^{(0,4)}_{ij} &= \hat{h}^{(0,4)}_{ij} + 2\chi^{(0,4)}_{ij} - 4\nabla^{-2}\chi^{(0,4),k}_{k(i,j)} + \left(\nabla^{-2}\chi^{(0,4),kl}_{kl} - \chi^{(0,4)}\right)\delta_{ij} \\ &\quad + \nabla^{-2}\left(\nabla^{-2}\chi^{(0,4),kl}_{kl} + \chi^{(0,4)}\right)_{,ij} \sim \eta^4\,.\end{aligned} \tag{B.66}$$

The quantity $\chi^{(0,4)}_{ij}$ is defined in Eq. (B.45), and here we have written $\chi^{(n,m)} \equiv \delta^{ij}\chi^{(n,m)}_{ij}$. In terms of irreducibly decomposed potentials, this quantity can be written as

$$\chi_{ij}^{(0,4)} = 2\left(-\psi_{,k}^{(0,2)}\delta_{ij} + E_{,ijk}^{(0,2)} + F_{(i,j)k}^{(0,2)} + \frac{1}{2}\hat{h}_{ij,k}^{(0,2)} + \delta x_{,ijk}^{(0,2)} + \delta x_{(i,j)k}^{(0,2)}\right)\left(\delta x^{(0,2),k} + \delta x^{(0,2)k}\right)$$
$$+ 2\left(-\psi^{(0,2)}\delta_{ik} + E_{,ik}^{(0,2)} + F_{(i,k)}^{(0,2)} + \frac{1}{2}\hat{h}_{ik}^{(0,2)} + \delta x_{,ik}^{(0,2)} + \delta x_{(i,k)}^{(0,2)}\right)\left(\delta x_{,j}^{(0,2),k} + \delta x_{,j}^{(0,2)k}\right)$$
$$+ 2\left(-\psi^{(0,2)}\delta_{jk} + E_{,jk}^{(0,2)} + F_{(j,k)}^{(0,2)} + \frac{1}{2}\hat{h}_{jk}^{(0,2)} + \delta x_{,jk}^{(0,2)} + \delta x_{(j,k)}^{(0,2)}\right)\left(\delta x_{,i}^{(0,2),k} + \delta x_{,i}^{(0,2)k}\right). \tag{B.67}$$

This completes the full set of transformations in the post-Newtonian sector. We note that the lowest order post-Newtonian metric potentials $\phi^{(0,2)}$ and $\psi^{(0,2)}$ transform as expected from post-Newtonian gravity [50]. As far as we are aware, the transformation of scalar, vector and tensor post-Newtonian potentials has not been calculated before. The above transformations are derived from our two-parameter formalism, but because there are only post-Newtonian (not cosmological or mixed-order) potentials and gauge generators in these transformations they also hold for one-parameter post-Newtonian gravity.

**Mixed-order scalar, vector and tensor potentials:** The scalar part of the mixed-order potentials, up to the order considered in the field equations in Section 6.4, $\mathcal{O}(\epsilon\eta^2)$, transform in the following way:

$$\tilde{\phi}^{(1,1)} = \phi^{(1,1)} + \phi_{,i}^{(0,2)}\left(\delta x^{(1,0),i} + \delta x^{(1,0)i}\right) \sim \epsilon\eta \tag{B.68}$$

$$\tilde{\phi}^{(1,2)} = \phi^{(1,2)} + \dot{\phi}^{(0,2)}\delta t^{(1,0)} + 2\phi^{(0,2)}\dot{\delta t}^{(1,0)} \sim \epsilon\eta^2 \tag{B.69}$$

$$\tilde{\psi}^{(1,1)} = \psi^{(1,1)} + \frac{1}{2}\left(\nabla^{-2}\chi_{ij}^{(1,1),ij} - \chi^{(1,1)}\right) \sim \epsilon\eta \tag{B.70}$$

$$\tilde{\psi}^{(1,2)} = \psi^{(1,2)} + \nabla^{-2}\left(\chi_{k[l}^{(1,2),k]l} + 2\mathcal{C}_{k[l|,m}^{,|k]}\mathcal{I}^{m,l}\right) \sim \epsilon\eta^2 \tag{B.71}$$

$$\tilde{B}^{(1,2)} = B^{(1,2)} + a\dot{\delta x}^{(1,1)} - \frac{1}{a}\delta t^{(1,2)} + \nabla^{-2}\chi_{i}^{(1,2),i} \sim \epsilon\eta^2 L_N \tag{B.72}$$

$$\tilde{E}^{(1,1)} = E^{(1,1)} + 2\delta x^{(1,1)} + \frac{1}{2}\nabla^{-2}(3\nabla^{-2}\chi_{ij}^{(1,1),ij} - \chi^{(1,1)}) \sim \epsilon\eta L_N^2 \tag{B.73}$$

$$\tilde{E}^{(1,2)} = E^{(1,2)} + 2\delta x^{(1,2)} + \frac{1}{2}\nabla^{-2}\left(\nabla^{-2}\left(3\chi_{kl}^{(1,2),kl} + 6\mathcal{C}_{kl,m}^{k}\mathcal{I}^{m,l} - 2\mathcal{C}_{k,l}^{k}\mathcal{I}^{m,l}\right) - \chi^{(1,2)}\right) \sim \epsilon\eta^2 L_N^2, \tag{B.74}$$

where we have used anti-symmetric square brackets that are defined by $2\mathcal{T}_{[ij]} \equiv$

$\mathcal{T}_{ij} - \mathcal{T}_{ji}$. The vector parts transform as

$$\tilde{B}_i^{(1,2)} = B_i^{(1,2)} + a\dot{\delta x}_i^{(1,1)} + \chi_i^{(1,2)} - \nabla^{-2}\chi_{j,i}^{(1,2),j} \sim \epsilon\eta^2 \tag{B.75}$$

$$\tilde{F}_i^{(1,1)} = F_i^{(1,1)} + 2\delta x_i^{(1,1)} + 2\nabla^{-2}\left(\chi_{ik}^{(1,1),k} - \nabla^{-2}\chi_{kj,i}^{(1,1),kj}\right) \sim \epsilon\eta L_N \tag{B.76}$$

$$\begin{aligned}\tilde{F}_i^{(1,2)} &= F_i^{(1,2)} + 2\delta x_i^{(1,2)} \\ &\quad - 2\nabla^{-2}\nabla^{-2}\left(2\chi_{k[i,l]}^{(1,2),kl} - 4\mathcal{C}_{k[i,l]m}^{,k}\mathcal{I}^{m,l} - \nabla^2\mathcal{C}_{ki,m}\mathcal{I}^{m,k} + \mathcal{C}_{kl,m}^{,kl}\mathcal{I}_{,i}^{m}\right) \sim \epsilon\eta^2 L_N\,,\end{aligned} \tag{B.77}$$

and the tensor parts transform as

$$\begin{aligned}\tilde{\hat{h}}_{ij}^{(1,1)} &= \hat{h}_{ij}^{(1,1)} + 2\chi_{ij}^{(1,1)} - 4\nabla^{-2}\chi_{k(i,j)}^{(1,1),k} + \nabla^{-2}\chi_{kl}^{(1,1),kl}\delta_{ij} - \chi^{(1,1)}\delta_{ij} \\ &\quad + \nabla^{-2}\nabla^{-2}\chi_{kl,ij}^{(1,1),kl} + \nabla^{-2}\chi_{,ij}^{(1,1)} \sim \epsilon\eta\end{aligned} \tag{B.78}$$

$$\begin{aligned}\tilde{\hat{h}}_{ij}^{(1,2)} &= \hat{h}_{ij}^{(1,2)} + 2\chi_{ij}^{(1,2)} - 4\nabla^{-2}\chi_{k(i,j)}^{(1,2),k} + \nabla^{-2}\chi_{kl}^{(1,2),kl}\delta_{ij} - \chi^{(1,2)}\delta_{ij} \\ &+ \nabla^{-2}\nabla^{-2}\chi_{kl,ij}^{(1,2),kl} + \nabla^{-2}\chi_{,ij}^{(1,2)} \\ &+ 4\nabla^{-2}\nabla^{-2}\Big(\nabla^2\mathcal{C}_{ij,mk}\mathcal{I}^{m,k} - \nabla^2\mathcal{C}_{k(i,j)m}\mathcal{I}^{m,k} - 2\mathcal{C}_{k(i,j)klm}\mathcal{I}^{m,l} \\ &\qquad - \nabla^2\mathcal{C}_{k(i|,m}^{,k}\mathcal{I}_{,|j)}^{m} + \mathcal{C}_{kl,mn}^{,k(l|}\mathcal{I}^{m,|n)}\delta_{ij}\Big) \\ &+ \nabla^{-2}\nabla^{-2}\left(-\nabla^2\mathcal{C}_{k,ml}^{k}\mathcal{I}^{m,l}\delta_{ij} + 2\mathcal{C}_{kl,mij}^{,k}\mathcal{I}^{m,l} + 2\mathcal{C}_{kl,m(i}^{,kl}\mathcal{I}_{,j)}^{m} + 2\mathcal{C}_{ij,mk}\mathcal{I}^{m,k}\right) \sim \epsilon\eta^2\,.\end{aligned} \tag{B.79}$$

Note that in the above equations we define $\nabla^{-2}f(\chi^{(n,m)})$ such that $\nabla^2[\nabla^{-2}f(\chi^{(n,m)})]$ is the leading order part of $f(\chi^{(n,m)})$ and no smaller, which strictly excludes higher order terms in $f(\chi^{(n,m)})$. In the above equations we have written $\chi_i^{(1,2)}$, $\chi_{ij}^{(1,2)}$ and $\chi_{ij}^{(1,1)}$ in terms of scalar, vector and tensor potentials and $\chi_{ij}^{(1,1)}$ in terms of $\mathcal{C}_{ij,m}$ and $\mathcal{I}^m$ in the following way

$$\begin{aligned}\chi_i^{(1,2)} &= \frac{1}{a}\phi^{(0,2)}\delta t_{,i}^{(1,0)} + a\left(-\psi^{0,2}\delta_{ij} + E_{,ij}^{(0,2)} + F_{(i,j)}^{(0,2)} + \frac{1}{2}\hat{h}_{ij}^{(0,2)} + \delta x_{,ij}^{(0,2)} + \delta x_{(i,j)}^{(0,2)}\right) \\ &\quad \times \left(\delta x^{(1,0),j} + \delta x^{(1,0)j}\right)^{\cdot} \\ &+ \left(B_{,i}^{(0,3)} + B_i^{(0,3)} - \frac{1}{2a}\delta t_{,i}^{(0,3)} + \frac{a}{2}\left(\delta x_{,i}^{(0,2)} + \delta x_i^{(0,2)}\right)^{\cdot}\right)_{,j}\left(\delta x^{(1,0),j} + \delta x^{(1,0)j}\right) \\ &+ \left(B_{,j}^{(1,0)} + B_j^{(1,0)} - \frac{1}{2a}\delta t_{,j}^{(1,0)} + \frac{a}{2}\left(\delta x_{,j}^{(1,0)} + \delta x_j^{(1,0)}\right)^{\cdot}\right)\left(\delta x^{(0,2),j} + \delta x^{(0,2)j}\right)_{,i}\end{aligned} \tag{B.80}$$

$$
\begin{aligned}
\chi_{ij}^{(1,2)} &= \left(-\psi^{(0,2)}\delta_{ij} + E_{,ij}^{(0,2)} + F_{(i,j)}^{(0,2)} + \frac{1}{2}\hat{h}_{ij}^{(0,2)} + \delta x_{,ij}^{(0,2)} + \delta x_{(i,j)}^{(0,2)}\right)^{\cdot} \delta t^{(1,0)} \\
&\quad + 2\frac{\dot{a}}{a}\left(-\psi^{(0,2)}\delta_{ij} + E_{,ij}^{(0,2)} + F_{(i,j)}^{(0,2)} + \frac{1}{2}\hat{h}_{ij}^{(0,2)} + 2\delta x_{,ij}^{(0,2)} + 2\delta x_{(i,j)}^{(0,2)}\right) \delta t^{(1,0)} \\
&\quad + \left(-\psi^{(0,2)}\delta_{ik} + E_{,ik}^{(0,2)} + F_{(i,k)}^{(0,2)} + \frac{1}{2}\hat{h}_{ik}^{(0,2)} + \delta x_{,ik}^{(0,2)} + \delta x_{(i,k)}^{(0,2)}\right) \left(\delta x^{(1,0),k} + \delta x^{(1,0)k}\right)_{,j} \\
&\quad + \left(-\psi^{(0,2)}\delta_{jk} + E_{,jk}^{(0,2)} + F_{(j,k)}^{(0,2)} + \frac{1}{2}\hat{h}_{jk}^{(0,2)} + \delta x_{,jk}^{(0,2)} + \delta x_{(j,k)}^{(0,2)}\right) \left(\delta x^{(1,0),k} + \delta x^{(1,0)k}\right)_{,i} \\
&\quad + \left(-\psi^{(1,0)}\delta_{ik} + E_{,ik}^{(1,0)} + F_{(i,k)}^{(1,0)} + \frac{1}{2}\hat{h}_{ik}^{(1,0)} + \delta x_{,ik}^{(1,0)} + \delta x_{(i,k)}^{(1,0)}\right) \left(\delta x^{(0,2),k} + \delta x^{(0,2)k}\right)_{,j} \\
&\quad + \left(-\psi^{(1,0)}\delta_{jk} + E_{,jk}^{(1,0)} + F_{(j,k)}^{(1,0)} + \frac{1}{2}\hat{h}_{jk}^{(1,0)} + \delta x_{,jk}^{(1,0)} + \delta x_{(j,k)}^{(1,0)}\right) \left(\delta x^{(0,2),k} + \delta x^{(0,2)k}\right)_{,i}
\end{aligned}
\tag{B.81}
$$

$$
\chi_{ij}^{(1,1)} = \mathcal{C}_{ij,k}\mathcal{I}^k \,, \tag{B.82}
$$

where we have defined

$$
\mathcal{C}_{ij,k} \equiv \left(-\psi^{(0,2)}\delta_{ij} + E_{,ij}^{(0,2)} + F_{(i,j)}^{(0,2)} + \frac{1}{2}\hat{h}_{ij}^{(0,2)} + \delta x_{,ij}^{(0,2)} + \delta x_{(i,j)}^{(0,2)}\right)_{,k} \sim \eta^2 L_N^{-1} \tag{B.83}
$$

$$
\mathcal{I}^k \equiv \delta x^{(1,0),k} + \delta x^{(1,0)k} \sim \epsilon\eta^{-1} L_N. \tag{B.84}
$$

This completes our treatment of gauge transformations of the metric tensor.

## B.5. Gauge-invariant metric perturbations

Having identified the transformation laws for all the relevant quantities, we are now in a position to construct gauge-invariant metric potentials, using a generalisation of Bardeen's method [32].

Our gauge invariant quantities reduce to the metric perturbations in Poisson gauge when $E = B = F_i = 0$ (with superscript indices omitted for brevity). Other gauge choices are possible; our choice of gauge however ensures that the field equations resemble Newtonian field equations, and is guaranteed to be well defined in both sectors of the theory.

We choose gauge generators, $\delta x, \delta x^i$ and $\delta t$, such that $\tilde{E} = \tilde{B} = \tilde{F}_i = 0$. These are then substituted back into the expressions for all of the transformed perturbations. The resulting expressions are automatically gauge invariant, since the original gauge transformations were written down in an arbitrary coordinate system. This implies that the newly constructed quantities cannot be gauge-dependent by definition.

We present the expressions resulting from this procedure for the cosmological sector, the post-Newtonian sector, and the mixed-order sector of the expansion. All quantities have been explicitly confirmed to be gauge invariant.

**Cosmological quantities:** We can form two scalar, one vector and one tensor gauge invariant quantities in the cosmological sector. These are given by:

$$\Phi^{(1,0)} = \phi^{(1,0)} - 2a\dot{B}^{(1,0)} - 2\dot{a}B^{(1,0)} + 2\dot{a}a\dot{E}^{(1,0)} + a^2\ddot{E}^{(1,0)} \tag{B.85}$$

$$\Psi^{(1,0)} = \psi^{(1,0)} + \dot{a}a\dot{E}^{(1,0)} - 2\dot{a}B^{(1,0)} \tag{B.86}$$

$$\mathbf{B}_i^{(1,0)} = B_i^{(1,0)} - \frac{a}{2}\dot{F}_i^{(1,0)} \tag{B.87}$$

$$\mathbf{h}_{ij}^{(1,0)} = \hat{h}_{ij}^{(1,0)}, \tag{B.88}$$

which are all $\mathcal{O}(\epsilon)$. These gauge invariant quantities are exactly those found by Bardeen in the context of standard cosmological perturbation theory [32].

**Post-Newtonian quantities:** We can form two scalar and one tensor gauge invariant quantities at $\mathcal{O}(\eta^2)$ in the post-Newtonian sector:

$$\Phi^{(0,2)} = \phi^{(0,2)} \tag{B.89}$$

$$\Psi^{(0,2)} = \psi^{(0,2)} \tag{B.90}$$

$$\mathbf{h}_{ij}^{(0,2)} = \hat{h}_{ij}^{(0,2)}. \tag{B.91}$$

There is one gauge invariant vector at $\mathcal{O}(\eta^3)$,

$$\mathbf{B}_i^{(0,3)} = B_i^{(0,3)} - \frac{a}{2}\dot{F}_i^{(0,2)}, \tag{B.92}$$

while there are two scalars and one tensor at $\mathcal{O}(\eta^4)$ ,

$$\begin{aligned}\Phi^{(0,4)} &= \phi^{(0,4)} - 4a\dot{B}^{(0,3)} - 4\dot{a}B^{(0,3)} + 4\dot{a}a\dot{E}^{(0,2)} \\ &\quad + 2a^2\ddot{E}^{(0,2)} - \phi^{(0,2)}{}_{,i}\left(E^{(0,2),i} + F^{(0,2)i}\right)\end{aligned} \tag{B.93}$$

$$\Psi^{(0,4)} = \psi^{(0,4)} - 4\dot{a}\left(B^{(0,3)} - \frac{a}{2}\dot{E}^{(0,2)}\right) + \frac{1}{2}\left(\nabla^{-2}\chi_{Lij}^{(0,4),ij} - \chi_L^{(0,4)}\right) \tag{B.94}$$

$$\begin{aligned}\mathbf{h}_{ij}^{(0,4)} &= \hat{h}_{ij}^{(0,4)} + 2\chi_{Lij}^{(0,4)} + \left(\nabla^{-2}\chi_{Lkl}^{(0,4),kl} - \chi_L^{(0,4)}\right)\delta_{ij} \\ &\quad + \nabla^{-2}\left(\nabla^{-2}\chi_{Lkl}^{(0,4),kl} + \chi_L^{(0,4)}\right)_{,ij} - 4\nabla^{-2}\chi_{Lk(i\ j)}^{(0,4),k},\end{aligned} \tag{B.95}$$

where $\chi_{Lij}^{(0,4)}$ is defined as

$$\begin{aligned}\chi_{Lij}^{(0,4)} = &- \left(-\psi_{,k}^{(0,2)}\delta_{ij} + \frac{1}{2}E_{,ijk}^{(0,2)} + \frac{1}{2}F_{(i,j)k}^{(0,2)} + \frac{1}{2}\hat{h}_{ij,k}^{(0,2)}\right)\left(E^{(0,2),k} + F^{(0,2)k}\right) \\ &- \left(-\psi^{(0,2)}\delta_{ik} + \frac{1}{2}E_{,ik}^{(0,2)} + \frac{1}{2}F_{(i,k)}^{(0,2)} + \frac{1}{2}\hat{h}_{ik}^{(0,2)}\right)\left(E_{,j}^{(0,2),k} + F_{,j}^{(0,2)k}\right) \\ &- \left(-\psi^{(0,2)}\delta_{jk} + \frac{1}{2}E_{,jk}^{(0,2)} + \frac{1}{2}F_{(j,k)}^{(0,2)} + \frac{1}{2}\hat{h}_{jk}^{(0,2)}\right)\left(E_{,i}^{(0,2),k} + F_{,i}^{(0,2)k}\right) . \end{aligned} \tag{B.96}$$

.

**Mixed-order quantities:** We can find two scalar and one tensor gauge invariant quantities at $\mathcal{O}(\epsilon\eta)$:

$$\Phi^{(1,1)} = \phi^{(1,1)} - \frac{1}{2}\phi_{,i}^{(0,2)}\left(E^{(1,0),i} + F^{(1,0)i}\right) \tag{B.97}$$

$$\Psi^{(1,1)} = \psi^{(1,1)} + \frac{1}{2}\left(\nabla^{-2}\chi_{Lij}^{(1,1),ij} - \chi_{L}^{(1,1)}\right) \tag{B.98}$$

$$\begin{aligned}\mathbf{h}_{ij}^{(1,1)} = &\hat{h}_{ij}^{(1,1)} + 2\chi_{Lij}^{(1,1)} - 4\nabla^{-2}\chi_{Lk(i,j)}^{(1,1),k} \\ &+ \nabla^{-2}\chi_{Lkl}^{(1,1),kl}\delta_{ij} - \chi_{L}^{(1,1)}\delta_{ij} + \nabla^{-2}\nabla^{-2}\chi_{Lkl,ij}^{(1,1),kl} + \nabla^{-2}\chi_{L,ij}^{(1,1)} .\end{aligned} \tag{B.99}$$

### B. Two-parameter gauge invariant metric potentials

There exist two gauge invariant scalars, one vector and one tensor at order $\mathcal{O}(\epsilon\eta^2)$ :

$$\begin{aligned}\Phi^{(1,2)} &= \phi^{(1,2)} + \dot{\phi}^{(0,2)}\left(aB^{(1,0)} - \frac{a^2}{2}\dot{E}^{(1,0)}\right)\\ &\quad + 2\phi^{(0,2)}\left(\dot{a}B^{(1,0)} + a\dot{B}^{(1,0)} - a\dot{a}\dot{E}^{(1,0)} - \frac{a^2}{2}\ddot{E}^{(1,0)}\right)\end{aligned} \tag{B.100}$$

$$\Psi^{(1,2)} = \psi^{(1,2)} + \nabla^{-2}\left(\chi^{(1,2),k]l}_{Lk[l} + 2\mathcal{C}^{,|k]}_{Lk[l|,m}\mathcal{I}^{m,l}_{L}\right) \tag{B.101}$$

$$\mathbf{B}^{(1,2)}_{i} = B^{(1,2)}_{i} - \frac{a}{2}\dot{F}^{(1,1)}_{i} + \chi^{(1,2)}_{Li} - \nabla^{-2}\chi^{(1,2),j}_{Lj,i} \tag{B.102}$$

$$\begin{aligned}\mathbf{h}^{(1,2)}_{ij} &= \hat{h}^{(1,2)}_{ij} + 2\chi^{(1,2)}_{Lij} - 4\nabla^{-2}\chi^{(1,2),k}_{Lk(i,j)} + \nabla^{-2}\chi^{(1,2),kl}_{Lkl}\delta_{ij}\\ &\quad - \chi^{(1,2)}_{L}\delta_{ij} + \nabla^{-2}\nabla^{-2}\chi^{(1,2),kl}_{Lkl,ij} + \nabla^{-2}\chi^{(1,2)}_{L,ij}\\ &\quad + 4\nabla^{-2}\nabla^{-2}\Bigg(\nabla^2\mathcal{C}_{Lij,mk}\mathcal{I}^{m,k}_{L} - \nabla^2\mathcal{C}_{Lk(i,j)m}\mathcal{I}^{m,k}_{L} - 2\mathcal{C}_{Lk(i,j)klm}\mathcal{I}^{m,l}_{L}\\ &\quad - \nabla^2\mathcal{C}^{,k}_{Lk(i|,m}\mathcal{I}^{m}_{L,|j)} + \mathcal{C}^{,k(l|}_{Lkl,mn}\mathcal{I}^{m,|n)}_{L}\delta_{ij}\Bigg)\\ &\quad + \nabla^{-2}\nabla^{-2}\Bigg( - \nabla^2\mathcal{C}^{,k}_{Lk,ml}\mathcal{I}^{m,l}_{L}\delta_{ij} + 2\mathcal{C}^{,k}_{Lkl,mij}\mathcal{I}^{m,l}_{L}\\ &\quad + 2\mathcal{C}^{,kl}_{Lkl,m(i}\mathcal{I}^{m}_{L,j)} + 2\mathcal{C}_{Lij,mk}\mathcal{I}^{m,k}_{L}\Bigg)\,.\end{aligned} \tag{B.103}$$

The fields $\chi_{Li}^{(1,2)}$, $\chi_{Lij}^{(1,2)}$ and $\chi_{Lij}^{(1,1)}$ are defined by

$$\begin{aligned}
\chi_{Li}^{(1,2)} &= \phi^{(0,2)}\left(B^{(1,0)} - \frac{a}{2}\dot{E}^{(1,0)}\right)_{,i} \\
&\quad - \frac{a}{2}\left(-\psi^{(0,2)}\delta_{ij} + \frac{1}{2}E^{(0,2)}_{,ij} + \frac{1}{2}F^{(0,2)}_{(i,j)} + \frac{1}{2}\hat{h}^{(0,2)}_{ij}\right)\left(E^{(1,0),j} + F^{(1,0)j}\right)\dot{\text{ffl}} \\
&\quad - \frac{1}{2}\left(\frac{1}{2}B^{(0,3)}_{,i} + B^{(0,3)}_{i} - \frac{a}{4}\dot{F}^{(0,2)}_{i}\right)_{,j}\left(E^{(1,0),j} + F^{(1,0)j}\right) \\
&\quad - \frac{1}{2}\left(\frac{1}{2}B^{(1,0)}_{,j} + B^{(1,0)}_{j} - \frac{a}{4}\dot{F}^{(1,0)}_{i}\right)\left(E^{(0,2),j} + \delta F^{(0,2)j}\right)_{,i} && \text{(B.104)} \\
\chi_{Lij}^{(1,2)} &= a\left(-\psi^{(0,2)}\delta_{ij} + \frac{1}{2}E^{(0,2)}_{,ij} + \frac{1}{2}F^{(0,2)}_{(i,j)} + \frac{1}{2}\hat{h}^{(0,2)}_{ij}\right)^{\cdot}\left(B^{(1,0)} - \frac{a}{2}\dot{E}^{(1,0)}\right) \\
&\quad + 2\dot{a}\left(-\psi^{(0,2)}\delta_{ij} + \frac{1}{2}\hat{h}^{(0,2)}_{ij}\right)\left(B^{(1,0)} - \frac{a}{2}\dot{E}^{(1,0)}\right) \\
&\quad - \frac{1}{2}\left(-\psi^{(0,2)}\delta_{ik} + \frac{1}{2}E^{(0,2)}_{,ik} + \frac{1}{2}F^{(0,2)}_{(i,k)} + \frac{1}{2}\hat{h}^{(0,2)}_{ik}\right)\left(E^{(1,0),k} + F^{(1,0)k}\right)_{,j} \\
&\quad - \frac{1}{2}\left(-\psi^{(0,2)}\delta_{jk} + \frac{1}{2}E^{(0,2)}_{,jk} + \frac{1}{2}F^{(0,2)}_{(j,k)} + \frac{1}{2}\hat{h}^{(0,2)}_{jk}\right)\left(E^{(1,0),k} + F^{(1,0)k}\right)_{,i} \\
&\quad - \frac{1}{2}\left(-\psi^{(1,0)}\delta_{ik} + \frac{1}{2}E^{(1,0)}_{,ik} + \frac{1}{2}F^{(1,0)}_{(i,k)} + \frac{1}{2}\hat{h}^{(1,0)}_{ik}\right)\left(E^{(0,2),k} + F^{(0,2)k}\right)_{,j} \\
&\quad - \frac{1}{2}\left(-\psi^{(1,0)}\delta_{jk} + \frac{1}{2}E^{(1,0)}_{,jk} + \frac{1}{2}F^{(1,0)}_{(j,k)} + \frac{1}{2}\hat{h}^{(1,0)}_{jk}\right)\left(E^{(0,2),k} + F^{(0,2)k}\right)_{,i} && \text{(B.105)} \\
\chi_{Lij}^{(1,1)} &= \mathcal{C}_{Lij,k}\mathcal{I}^{k}_{L}\,, && \text{(B.106)}
\end{aligned}$$

where $\mathcal{C}_{Lij,k}$ and $\mathcal{I}^{k}_{L}$ are given by

$$\begin{aligned}
\mathcal{C}_{Lij,k} &\equiv \left(-\psi^{(0,2)}\delta_{ij} + \frac{1}{2}E^{(0,2)}_{,ij} + \frac{1}{2}F^{(0,2)}_{(i,j)} + \frac{1}{2}\hat{h}^{(0,2)}_{ij}\right)_{,k} && \text{(B.107)} \\
\mathcal{I}^{k}_{L} &\equiv -\frac{1}{2}\left(E^{(1,0),k} + F^{(1,0)k}\right)\,. && \text{(B.108)}
\end{aligned}$$

These are all the gauge invariant quantities that can be constructed for metric perturbations.

It is immediately clear that there are multiple gauge-invariant quantities in the theory, including the scalar Newtonian and post-Newtonian potentials $\phi^{(0,2)}$ and $\psi^{(0,2)}$, as well as the leading order tensor fields $\hat{h}^{(1,0)}_{ij}$ and $\hat{h}^{(0,2)}_{ij}$. The first two correspond to the gravitational potential in the Newton-Poisson equation, and are therefore expected to be gauge-invariant. The last two show that the leading-order tensor fluctuations are gauge-invariant in both the cosmological and post-Newtonian sectors. It is interesting to note that the form of the gauge-invariant quantities $\Phi^{(1,0)}$ and $\Phi^{(0,4)}$ differ by a single term: $-\frac{1}{2}\phi^{(0,2)}{}_{,i}\,(E^{(0,2),i} + F^{(0,2)i})$, which is quadratic. It is not possible for the cosmological gauge invariant quantity $\Phi^{(1,0)}$ to contain a term of this form, as it would be higher order, at $\mathcal{O}(\epsilon^2)$.

# C. Two-parameter field equations

In this appendix, we present the two-parameter field equations, both in their gauge unfixed form, and in terms of gauge invariant variables. These equations were first derived in the dust case in [125], and extended to the case with background pressure, radiation and a cosmological constant in [127]. We repeat the treatment given in that paper here for the ease of the reader.

## C.1. Field equations without gauge fixing

This section contains the field equations in terms of the variables introduced in Section 6.2, with the choice of relations between $\epsilon$, $\eta$, $L_C$ and $L_N$ given in Eq. (6.29).

### C.1.1. Background-order potentials

The leading-order 00-field equation is of order $\mathcal{O}(\eta^2 L_N^{-2})$ and is given by

$$3\frac{\ddot{a}}{a} + \frac{1}{2a^2}\nabla^2 h_{00}^{(0,2)} = -4\pi\left(\rho^{(0,0)} + \rho^{(0,2)} + 3P^{(0,0)}\right) + \Lambda^{(0,0)} . \tag{C.1}$$

Note that $a \sim 1$ and $\ddot{a} \sim 1/L_C^2$, as the time variation of $a(t)$ is over cosmological scales. This equation is a combination of the Raychaudhuri equation from Friedmann cosmology, and the Newton-Poisson equation from post-Newtonian gravity. The leading-order contribution to the trace of the $ij$-field equation at $\mathcal{O}(\eta^2 L_N^{-2})$ is given by

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{6a^2}\left(\nabla^2 h_{ii}^{(0,2)} - h_{ij,ij}^{(0,2)}\right) = \frac{8\pi}{3}\left(\rho^{(0,2)} + \rho^{(0,0)}\right) + \frac{1}{3}\Lambda^{(0,0)} . \tag{C.2}$$

This equation is a combination of the Friedmann constraint equation and the Newton-Poisson equation for the trace of the post-Newtonian potential $h_{ii}^{(0,2)}$. At the same order, the trace-free part of the $ij$-field equations is

$$D_{ij}\left(h_{00}^{(0,2)} - h_{kk}^{(0,2)}\right) + 2h_{k\langle i,j\rangle k}^{(0,2)} - \nabla^2 h_{\langle ij\rangle}^{(0,2)} = 0\,. \tag{C.3}$$

This equation is the same as given in Ref. [125] because neither the cosmological constant nor radiation contribute trace-free components. The other equations contain contributions from both radiation and the cosmological constant.

### C.1.2. Vector potentials

The $0i$-field equations give the governing equations for the vector gravitational potentials. The leading-order contribution is $\mathcal{O}(\eta^3 L_N^{-2})$ and is given by

$$\nabla^2 h_{0i}^{(0,3)} - h_{0j,ij}^{(0,3)} - a\dot{h}_{ij,j}^{(0,2)} + a\dot{h}_{jj,i}^{(0,2)} + 2\dot{a}h_{00,i}^{(0,2)} = 16\pi a^2\left(\rho^{(0,0)} + \rho^{(0,2)} + P^{(0,0)}\right)v_i^{(0,1)}\,. \tag{C.4}$$

This is the equation for the small-scale post-Newtonian vector potential, responsible for phenomena such as the Lense-Thirring effect, and is the one studied in Ref. [132]. Interestingly, this field equation implies the gravitomagnetic potential is $\sim 100$ times larger than second-order perturbation theory predicts [125, 152]. The next-to-leading-order $0i$-field equation occurs at $\mathcal{O}(\eta^4 L_N^{-2})$, and is given by

$$\begin{aligned}
&\nabla^2\left(h_{0i}^{(1,0)} + h_{0i}^{(1,2)}\right) - \left(h_{0j}^{(1,0)} + h_{0j}^{(1,2)}\right)_{,ij} - h_{0j}^{(1,0)}h_{00,ij}^{(0,2)} - a\left(h_{ij}^{(1,0)} + h_{ij}^{(1,1)}\right)^{\cdot}_{,j}\\
&+ 2\dot{a}\left(h_{00}^{(1,0)} + h_{00}^{(1,1)}\right)_{,i} - 2h_{0i}^{(1,0)}\left(2\dot{a}^2 + a\ddot{a}\right) + a\left(h_{jj}^{(1,0)} + h_{jj}^{(1,1)}\right)^{\cdot}_{,i}\\
&= 8\pi a^2\left(2\left(\rho^{(0,0)} + \rho^{(0,2)} + P^{(0,0)}\right)v_i^{(1,0)} + 2\rho^{(1,1)}v_i^{(0,1)} + \left(\rho^{(0,0)} + \rho^{(0,2)} + 3P^{(0,0)}\right)h_{0i}^{(1,0)}\right)\\
&\quad - 2a^2\Lambda^{(0,0)}h_{0i}^{(1,0)}\,,
\end{aligned} \tag{C.5}$$

This equation is the governing equation for the large-scale vector potentials. It is more complicated than Eq. (C.4), and shows that non-linear gravitational effects could potentially source the growth of large-scale vector potentials at late times. This equation can also be seen to have contributions from radiation and the cosmological constants, unlike Eq. (C.4).

### C.1.3. Higher-order scalar potentials

The next-to-leading-order 00-field equation occurs at $\mathcal{O}(\eta^3 L_N^{-2})$, and given by

$$\nabla^2 h_{00}^{(1,1)} = -8\pi a^2 \rho^{(1,1)} \,. \tag{C.6}$$

This is another version of the Newton-Poisson equation, and is sourced only by a mixed-order matter energy density, $\rho^{(1,1)}$. The governing equations for the cosmological potentials $h_{00}^{(1,0)}$ and $h_{ii}^{(1,0)}$ occur along with post-Newtonian and mixed-order potentials at $\mathcal{O}(\eta^4 L_N^{-2})$ (as was the case for the vector potentials considered above). The 00-field equation at this order gives

$$\begin{aligned}
&\nabla^2 \left(h_{00}^{(1,0)} + h_{00}^{(1,2)} + \frac{1}{2} h_{00}^{(0,4)}\right) + \frac{1}{2}\left(\nabla h_{00}^{(0,2)}\right)^2 + a^2 \left(h_{ii}^{(0,2)} + h_{ii}^{(1,0)}\right)^{\cdot\cdot} \\
&- 2\left[a\left(h_{0i}^{(0,3)} + h_{0i}^{(1,0)}\right)_{,i}\right]^{\cdot} + 2a\dot{a}\left(h_{ii}^{(0,2)} + h_{ii}^{(1,0)}\right)^{\cdot} - \frac{1}{2} h_{00,i}^{(0,2)} \left(2h_{ij,j}^{(0,2)} - h_{jj,i}^{(0,2)}\right) \\
&- h_{00,ij}^{(0,2)} \left(h_{ij}^{(1,0)} + h_{ij}^{(0,2)}\right) + 3a\dot{a}\left(h_{00}^{(0,2)} + h_{00}^{(1,0)}\right)^{\cdot} \\
&= -8\pi a^2 \Big[\rho^{(1,0)} + \rho^{(1,2)} + \frac{1}{2}\rho^{(0,4)} - \left(\rho^{(0,0)} + \rho^{(0,2)} + 3P^{(0,0)}\right)\left(h_{00}^{(1,0)} + h_{00}^{(0,2)}\right) \\
&\qquad + 3\Big(P^{(1,0)} + P^{(1,2)} + \frac{1}{2}P^{(0,4)}\Big)\Big] \\
&- 16\pi a^2 \left(v_i^{(0,1)}\right)^2 \left(\rho^{(0,0)} + \rho^{(0,2)} + P^{(0,0)}\right) - 2a^2 \Lambda^{(0,0)} \left(h_{00}^{(0,2)} + h_{00}^{(1,0)}\right) \,.
\end{aligned} \tag{C.7}$$

This equation can be seen to have additional sources due to the presence of radiation and a cosmological constant, compared to the corresponding equation in the presence of dust only [125]. The next non-trivial order in the $ij$-field equation is at $\mathcal{O}(\eta^3 L_N^{-2})$. Its trace gives

$$\nabla^2 h_{ii}^{(1,1)} - h_{ij,ij}^{(1,1)} = -16\pi a^2 \rho^{(1,1)} \,, \tag{C.8}$$

and its trace-free part is given below. Similarly the $ij$-field equation at $\mathcal{O}(\eta^4 L_N^{-2})$ can also be split into its trace and trace-free parts. The trace of this equation gives

$$\begin{aligned}
&\left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)\left(h_{ij}^{(1,0)} + h_{ij}^{(1,2)} + \frac{1}{2}h_{ij}^{(0,4)}\right) \\
&- \left(2\dot{a}^2 + a\ddot{a}\right)\left(h_{ii}^{(1,0)} + h_{ii}^{(0,2)} + 3h_{00}^{(1,0)} + 3h_{00}^{(0,2)}\right) \\
&+ 4\dot{a}\left(h_{0i}^{(1,0)} + h_{0i}^{(0,3)}\right)_{,i} - 2a\dot{a}\left(h_{ii}^{(1,0)} + h_{ii}^{(0,2)}\right)^{\cdot} \\
&= -16\pi a^2\left[\rho^{(1,0)} + \frac{1}{2}\rho^{(0,4)} + \rho^{(1,2)} + \left(\rho^{(0,0)} + \rho^{(0,2)} + P^{(0,0)}\right)\left(v_i^{(0,1)}\right)^2\right] \\
&- 4\pi a^2\Big[\left(\rho^{(0,0)} + \rho^{(0,2)} - P^{(0,0)}\right)\left(h_{ii}^{(0,2)} + h_{ii}^{(1,0)}\right) \\
&- \left(\rho^{(0,0)} + \rho^{(0,2)} + 3P^{(0,0)}\right)\left(h_{00}^{(0,2)} + h_{00}^{(1,0)}\right)\Big] \\
&- a^2\Lambda^{(0,0)}\left[h_{00}^{(0,2)} + h_{00}^{(1,0)} + h_{ii}^{(0,2)} + h_{ii}^{(1,0)}\right] + \mathcal{A}\,,
\end{aligned} \tag{C.9}$$

where we have simplified using Eq. (C.7), and where $\mathcal{A}$ is given by

$$\begin{aligned}
\mathcal{A} \equiv\ & \frac{3}{4}\left(h_{ij,k}^{(0,2)}\right)^2 + h_{ij,j}^{(0,2)}\left(h_{kk,i}^{(0,2)} - h_{ik,k}^{(0,2)}\right) - \frac{1}{2}h_{ij,k}^{(0,2)}h_{ik,j}^{(0,2)} \\
& - \frac{1}{4}h_{ii,j}^{(0,2)}h_{kk,j}^{(0,2)} + \frac{1}{2}\nabla^2 h_{00}^{(0,2)}\left(h_{00}^{(1,0)} + h_{00}^{(0,2)}\right) \\
& + \frac{1}{2}\left(h_{00,ij}^{(0,2)} + \nabla^2 h_{ij}^{(0,2)}\right)\left(h_{ij}^{(1,0)} + h_{ij}^{(0,2)}\right) + \left(\frac{1}{2}h_{ii,jk}^{(0,2)} - h_{ij,ik}^{(0,2)}\right)\left(h_{jk}^{(0,2)} + h_{jk}^{(1,0)}\right).
\end{aligned} \tag{C.10}$$

This equation includes new source terms due to the radiation and cosmological constant. The trace-free part of this equation is presented below.

### C.1.4. Tensor potentials

The next-to-leading-order trace-free $ij$-field equation occurs at $\mathcal{O}(\eta^3 L_N^{-2})$, and is given by

$$D_{ij}\left(h_{00}^{(1,1)} - h_{kk}^{(1,1)}\right) + 2h_{k\langle i,j\rangle k}^{(1,1)} - \nabla^2 h_{\langle ij\rangle}^{(1,1)} = 0\,. \tag{C.11}$$

Finally, the trace-free part of the $ij$-field equation at $\mathcal{O}(\eta^4 L_N^{-2})$ is given by

$$\nabla^2\left(h^{(1,0)}_{\langle ij\rangle}+h^{(1,2)}_{\langle ij\rangle}+\frac{1}{2}h^{(0,4)}_{\langle ij\rangle}\right)-D_{ij}\left(h^{(1,0)}_{00}+h^{(1,2)}_{00}+\frac{1}{2}h^{(0,4)}_{00}-h^{(1,0)}_{kk}-h^{(1,2)}_{kk}-\frac{1}{2}h^{(0,4)}_{kk}\right)$$
$$-2\left(h^{(1,0)}_{k\langle i}+h^{(1,2)}_{k\langle i}+\frac{1}{2}h^{(0,4)}_{k\langle i}\right)_{,j\rangle k}-a^2\left(h^{(1,0)}_{\langle ij\rangle}+h^{(0,2)}_{\langle ij\rangle}\right)^{\cdot\cdot}$$
$$-2\left(2\dot{a}^2+a\ddot{a}\right)\left(h^{(1,0)}_{\langle ij\rangle}+h^{(0,2)}_{\langle ij\rangle}\right)-3a\dot{a}\left(h^{(1,0)}_{\langle ij\rangle}+h^{(0,2)}_{\langle ij\rangle}\right)^{\cdot}+\frac{2}{a}\left[a^2\left(h^{(1,0)}_{0\langle i}+h^{(0,3)}_{0\langle i}\right)\right]^{\cdot}_{,j\rangle}$$
$$=-8\pi a^2\Big[\left(\rho^{(0,0)}+\rho^{(0,2)}-P^{(0,0)}\right)\left(h^{(0,2)}_{\langle ij\rangle}+h^{(1,0)}_{\langle ij\rangle}\right)+$$
$$2\left(\rho^{(0,0)}+\rho^{(0,2)}+P^{(0,0)}\right)v^{(0,1)}_{\langle i}v^{(0,1)}_{j\rangle}\Big]-2a^2\Lambda^{(0,0)}\left(h^{(0,2)}_{\langle ij\rangle}+h^{(1,0)}_{\langle ij\rangle}\right)+\mathcal{B}_{ij}\,, \tag{C.12}$$

where

$$\begin{aligned}\mathcal{B}_{ij}\equiv{}&\frac{1}{2}h^{(0,2)}_{00,\langle i|}h^{(0,2)}_{00,|j\rangle}+\frac{1}{2}h^{(0,2)}_{kl,\langle i|}h^{(0,2)}_{kl,|j\rangle}+D_{ij}h^{(0,2)}_{00}\left(h^{(1,0)}_{00}+h^{(0,2)}_{00}\right)\\&+\frac{1}{2}\left(h^{(0,2)}_{00,k}+2h^{(0,2)}_{kl,l}-h^{(0,2)}_{ll,k}\right)\left(h^{(0,2)}_{\langle ij\rangle,k}-2h^{(0,2)}_{k\langle i,j\rangle}\right)\\&+\left(D_{ij}h^{(0,2)}_{kl}+h^{(0,2)}_{\langle ij\rangle,kl}-2h^{(0,2)}_{k\langle i,j\rangle l}\right)\left(h^{(1,0)}_{kl}+h^{(0,2)}_{kl}\right)+h^{(0,2)}_{\langle i|k,l}\left(h^{(0,2)}_{|j\rangle k,l}-h^{(0,2)}_{|j\rangle l,k}\right).\end{aligned} \tag{C.13}$$

This completes the full set of field equations, to the order at which we require them.

## C.2. Field equations in gauge-invariant variables

This appendix contains the field equations in terms of the gauge-invariant variables from Section B.2 and [125]. The choice of relations between $\epsilon$, $\eta$, $L_C$ and $L_N$ is again the same as those given in Eq. (6.29).

### C.2.1. Background-order potentials

The trace-free part of the $ij$-equations at $\mathcal{O}(\eta^2 L_N^{-2})$ gives

$$D_{ij}\left(\Phi^{(0,2)}+\Psi^{(0,2)}\right)-\frac{1}{2}\nabla^2\mathbf{h}^{(0,2)}_{ij}=0\,, \tag{C.14}$$

which implies

$$\Phi^{(0,2)}=-\Psi^{(0,2)}\qquad\text{and}\qquad\mathbf{h}^{(0,2)}_{ij}=0\,. \tag{C.15}$$

The 00-field equation at $\mathcal{O}(\eta^2 L_N^{-2})$ can be written as

$$\frac{\ddot{a}}{a} + \frac{1}{6a^2}\nabla^2\Phi^{(0,2)} = -\frac{4\pi}{3}\left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} + 3\mathbf{P}^{(0,0)}\right) + \frac{1}{3}\boldsymbol{\Lambda}\,, \tag{C.16}$$

and the trace of the $ij$-equation at $\mathcal{O}(\eta^2 L_N^{-2})$ gives

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3a^2}\nabla^2\Phi^{(0,2)} = \frac{8\pi}{3}\left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)}\right) + \frac{1}{3}\boldsymbol{\Lambda}\,, \tag{C.17}$$

where we have substituted in the results from Eq. (C.15). These equations govern the leading-order part of the gravitational field, at $\mathcal{O}(\eta^2 L_N^{-2})$.

### C.2.2. Vector potentials

The $0i$-field equations at order $\mathcal{O}(\eta^3 L_N^{-2})$ give

$$\nabla^2\mathbf{B}_i^{(0,3)} + 2\left(a\dot{\Phi}^{(0,2)} + \dot{a}\Phi^{(0,2)}\right)_{,i} = 16\pi a^2\big(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} + \mathbf{P}^{(0,0)}\big)\mathbf{v}_i^{(0,1)}\,. \tag{C.18}$$

Although $\mathbf{B}_i^{(0,3)}$ is purely a divergenceless vector Eq. (C.18) has a divergenceless vector and scalar part, which can be separated out with a derivative. At $\mathcal{O}(\eta^4 L_N^{-2})$ the $0i$-field equations give

$$\begin{aligned}
&\nabla^2(\mathbf{B}_i^{(1,0)} + \mathbf{B}_i^{(1,2)} + 2\left(a\left(\Phi^{(1,1)} - \Psi^{(1,0)}\right)^{\cdot} + \dot{a}\left(\Phi^{(1,1)} + \Phi^{(1,0)}\right)\right)_{,i} \\
&- 2\left(2\dot{a}^2 + a\ddot{a}\right)\mathbf{B}_i^{(1,0)} - \mathbf{B}_j^{(1,0)}\Phi_{,ij}^{(0,2)} \\
&= 8\pi a^2\Bigg(2\big(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} + \mathbf{P}^{(0,0)}\big)\mathbf{v}_i^{(1,0)} + 2\boldsymbol{\rho}^{(1,1)}\mathbf{v}_i^{(0,1)} \\
&+ \big(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} + 3\mathbf{P}^{(0,0)}\big)\mathbf{B}_i^{(1,0)}\Bigg) - 2a^2\boldsymbol{\Lambda}\mathbf{B}_i^{(1,0)}\,,
\end{aligned} \tag{C.19}$$

which can also be split into scalar and divergenceless vector part using a derivative. The reader may note that the quadratic term, which includes the lower-order potential $\Phi^{(0,2)}$, does not source the vector part of Eq. (C.19).

### C.2.3. Higher-order scalar potentials

The 00-field equation and the trace of the $ij$-field equation at $\mathcal{O}(\epsilon\eta L_N^{-2})$ gives

$$\nabla^2\Phi^{(1,1)} = -8\pi a^2\boldsymbol{\rho}^{(1,1)}\,, \tag{C.20}$$

which implies

$$\Phi^{(1,1)} = -\Psi^{(1,1)} \,. \tag{C.21}$$

Using the 00-field equation at $\mathcal{O}(\eta^4 L_N^{-2})$ is

$$\begin{aligned}
&\nabla^2\left(\Phi^{(1,0)} + \frac{1}{2}\Phi^{(0,4)} + \Phi^{(1,2)}\right) + \left(\nabla\Phi^{(0,2)}\right)^2 + 3a\dot{a}\left(3\Phi^{(0,2)} + \Phi^{(1,0)} - 2\Psi^{(1,0)}\right)^{\cdot} \\
&+ 3a^2\left(\Phi^{(0,2)} - \Psi^{(1,0)}\right)^{\cdot\cdot} - \nabla^2\Phi^{(0,2)}\left(\Phi^{(0,2)} - \Psi^{(1,0)}\right) - \frac{1}{2}\Phi^{(0,2)}_{,ij}\mathbf{h}^{(1,0)}_{ij} \\
&= -8\pi a^2\Bigg[\boldsymbol{\rho}^{(1,0)} + \boldsymbol{\rho}^{(1,2)} + \frac{1}{2}\boldsymbol{\rho}^{(0,4)} - \left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} + 3\mathbf{P}^{(0,0)}\right)\left(\Phi^{(1,0)} + \Phi^{(0,2)}\right) \\
&+ 3\left(\mathbf{P}^{(1,0)} + \mathbf{P}^{(1,2)} + \frac{1}{2}\mathbf{P}^{(0,4)}\right)\Bigg] \\
&- 16\pi a^2\left(\mathbf{v}^{(0,1)}_i\right)^2\left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} + \mathbf{P}^{(0,0)}\right) - 2a^2\boldsymbol{\Lambda}\left(\Phi^{(0,2)} + \Phi^{(1,0)}\right) \,,
\end{aligned} \tag{C.22}$$

while the trace of the $ij$-field equation at $\mathcal{O}(\eta^4 L_N^{-2})$ gives

$$\begin{aligned}
&-2\nabla^2\left(\Psi^{(1,0)} + \Psi^{(1,2)} + \frac{1}{2}\Psi^{(0,4)}\right) - 3\left(2\dot{a}^2 + a\ddot{a}\right)\left(\Phi^{(1,0)} - \Psi^{(1,0)} + 2\Phi^{(0,2)}\right) \\
&+ 6\dot{a}a\left(\Psi^{(1,0)} - \Phi^{(0,2)}\right)^{\cdot} \\
&= -16\pi a^2\left[\boldsymbol{\rho}^{(1,0)} + \frac{1}{2}\boldsymbol{\rho}^{(0,4)} + \boldsymbol{\rho}^{(1,2)} + \left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} + \mathbf{P}^{(0,0)}\right)\left(\mathbf{v}^{(0,1)}_i\right)^2\right] \\
&- 4\pi a^2\Bigg[2\Phi^{(0,2)}\left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} - 3\mathbf{P}^{(0,0)}\right) - \left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)}\right)\left(\Phi^{(1,0)} + 3\Psi^{(1,0)}\right) \\
&+ 3\mathbf{P}^{(0,0)}\left(\Psi^{(1,0)} - \Phi^{(1,0)}\right)\Bigg] - a^2\boldsymbol{\Lambda}\left[4\Phi^{(0,2)} + \Phi^{(1,0)} - 3\Psi^{(1,0)}\right] + \mathcal{A} \,,
\end{aligned} \tag{C.23}$$

where

$$\mathcal{A} \equiv \nabla^2\Phi^{(0,2)}\left(3\Phi^{(0,2)} + \frac{1}{2}\Phi^{(1,0)} - \frac{5}{2}\Psi^{(1,0)}\right) + \frac{3}{2}\left(\nabla\Phi^{(0,2)}\right)^2 + \frac{1}{2}\Phi^{(0,2)}_{,ij}\mathbf{h}^{(1,0)}_{ij} \,. \tag{C.24}$$

These are all of the scalar equations that exist up to $\mathcal{O}(\eta^4 L_N^{-2})$.

### C.2.4. Tensor potentials

The trace-free part of the $ij$-field equation at $\mathcal{O}(\epsilon\eta L_N^{-2})$ is

$$D_{ij}\left(\Phi^{(1,1)} + \Psi^{(1,1)}\right) - \frac{1}{2}\nabla^2 \mathbf{h}_{ij}^{(1,1)} = 0\,, \tag{C.25}$$

which implies

$$\Phi^{(1,1)} = -\Psi^{(1,1)} \qquad \text{and} \qquad \mathbf{h}_{ij}^{(1,1)} = 0\,. \tag{C.26}$$

The reader may note that, unlike $\Psi^{(0,2)}$ and $\Phi^{(0,2)}$, the first of these conditions has already been given by the 00-field equation and the trace of the $ij-$field equations (C.21). Finally, the $\mathcal{O}(\eta^4 L_N^{-2})$ part of the $ij$-field equation can be used to write

$$\begin{aligned}
&- D_{ij}\left(\Phi^{(1,0)} + \Phi^{(1,2)} + \frac{1}{2}\Phi^{(0,4)} + \Psi^{(1,0)} + \Psi^{(1,2)} + \frac{1}{2}\Psi^{(0,4)}\right)\\
&+ \frac{1}{2}\nabla^2\left(\mathbf{h}_{ij}^{(1,0)} + \mathbf{h}_{ij}^{(1,2)} + \frac{1}{2}\mathbf{h}_{ij}^{(0,4)}\right)\\
&+ \frac{2}{a}\left[a^2\left(\mathbf{B}_{(i,j)}^{(0,3)} + \mathbf{B}_{(i,j)}^{(1,0)}\right)\right]^{\cdot} - \left(2\dot{a}^2 + a\ddot{a}\right)\mathbf{h}_{ij}^{(1,0)} - \frac{3}{2}a\dot{a}\dot{\mathbf{h}}_{ij}^{(1,0)} - \frac{1}{2}a^2\ddot{\mathbf{h}}_{ij}^{(1,0)}\\
&= -4\pi a^2\left[\left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} - \mathbf{p}^{(0,0)}\right)\mathbf{h}_{ij}^{(1,0)} + 4\left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} + \mathbf{p}^{(0,0)}\right)\mathbf{v}_{\langle i}^{(0,1)}\mathbf{v}_{j\rangle}^{(0,1)}\right]\\
&\qquad - a^2\boldsymbol{\Lambda}\mathbf{h}_{ij}^{(1,0)} + \mathcal{B}_{ij}\,,
\end{aligned} \tag{C.27}$$

where

$$\mathcal{B}_{ij} \equiv \quad D_{ij}\Phi^{(0,2)}\left(2\Phi^{(0,2)} + \Phi^{(1,0)} - \Psi^{(1,0)}\right) + \Phi_{,\langle i}^{(0,2)}\Phi_{,j\rangle}^{(0,2)} - \Phi_{,k\langle i}^{(0,2)}\mathbf{h}_{j\rangle k}^{(1,0)}\,, \tag{C.28}$$

and where we have used Eq. (C.26). Note that, unlike standard cosmological perturbation theory, these equations do not imply $\Phi^{(1,0)} = -\Psi^{(1,0)}$ or $\mathbf{h}_{ij}^{(1,0)} = 0$, and that scalar, vector and tensor modes do not decouple at linear order in cosmological perturbations. This completes the full set of field equations in terms of our gauge-invariant variables, up to the order in perturbations that we wish to consider here.

# D. Derivation of the higher order conservation equations

This appendix contains the derivation of the cosmological Euler equations that were withheld from Section 6.6. These equations are found by differentiating the constraint equations (6.49) and (6.50), and by using the evolution equations (6.48) and (6.52). As these equations contain post-Newtonian, mixed and cosmological terms, the reader must take into account the fact that spatial derivatives act differently on different types of perturbed quantities, as explained in Section 6.5.2. For example, carrying out calculations involving conformal time derivatives of the $\mathcal{O}(\eta^4)$ Einstein equations requires some knowledge of the $\mathcal{O}(\eta^5)$ equations.

## D.1. Derivation of the $\mathcal{O}(\eta^5)$ continuity equation

The conformal time derivative of Eq. (6.49) is given by

$$\begin{aligned}\frac{1}{3}\nabla\psi' - \mathcal{H}\psi'' - \mathcal{H}'\psi' - 2\mathcal{H}\mathcal{H}'\phi - \mathcal{H}^2\phi' = {} & \frac{4\pi a^2}{3}\big(\delta\rho' + \delta\rho'_{\mathrm{eff}} + 2\mathcal{H}(\delta\rho + \delta\rho_{\mathrm{eff}})\big) \\ & - \frac{16\pi a^2}{3}\big(\delta\rho_{\mathrm{S}}\psi' + \psi(\delta\rho'_{\mathrm{S}} + 2\mathcal{H}\delta\rho_{\mathrm{S}})\big) \\ & + \frac{1}{3}\mathcal{D}^{ij}\mathrm{U}_{\mathrm{S}}\mathrm{h}'_{ij} + \frac{1}{3}\mathcal{D}^{ij}\mathrm{U}'_{\mathrm{S}}\mathrm{h}_{ij} \ . \end{aligned} \tag{D.1}$$

Substituting in for $\psi''$ using Eq. (6.48) yields

$$\begin{aligned}\nabla^2(\psi' + \mathcal{H}\phi) = {} & - 3(\mathcal{H}^2 - \mathcal{H}')\psi' + 4\pi a^2\big(\delta\rho' + 3\mathcal{H}(\delta\rho + \delta\mathrm{p})\big) \\ & + 4\pi a^2\big(\delta\rho'_{\mathrm{eff}} + 3\mathcal{H}(\delta\rho_{\mathrm{eff}} + \delta\mathrm{p}_{\mathrm{eff}})\big) + 8\pi a^2\mathcal{H}\delta\rho_{\mathrm{S}}\phi \\ & + \mathcal{D}^{ij}\mathrm{U}'_{\mathrm{S}}\mathrm{h}_{ij} + \mathcal{D}^{ij}\mathrm{U}_{\mathrm{S}}\mathrm{h}'_{ij} + 2\mathcal{H}\mathcal{D}^{ij}\mathrm{U}_{\mathrm{S}}\mathrm{h}_{ij} \\ & - 16\pi a^2\big(\delta\rho_{\mathrm{S}}\psi' + \psi(\delta\rho'_{\mathrm{S}} + 2\mathcal{H}\delta\rho_{\mathrm{S}})\big) - 8\pi a^2\mathcal{H}\delta\rho_{\mathrm{S}}\psi \ . \end{aligned} \tag{D.2}$$

On the left-hand side of this equation, $\nabla^2(\psi' + \mathcal{H}\phi)$ is obtained from the divergence of the vector Einstein equation at $\mathcal{O}(\eta^5)$. Expanding this equation gives

$$\begin{aligned}
\nabla^2(\psi' + \mathcal{H}\phi) = \; & 4\pi a^2 \partial^i \big(\mathrm{v}_{\mathrm{N}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\mathrm{U_S}\big) - 4\pi a^2 \partial^i \big(\mathrm{v}_{\mathrm{M}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\big) \\
& + 8\pi a^2 \partial^i \big(\mathrm{v}_{\mathrm{N}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\big)\psi - 4\pi a^2 \big(\bar\rho + \bar p + \delta\rho_\mathrm{S}\big)\partial^i \mathrm{v}_{\mathrm{C}i} \\
& + 2\mathcal{H}\mathrm{U_S}\nabla^2\mathrm{U_S} - \mathrm{U_S'}\nabla^2\mathrm{U_S} + 2\mathcal{H}\phi\nabla^2\mathrm{U_S} - \psi'\nabla^2\mathrm{U_S} - 2\mathrm{U_S}\nabla^2\mathrm{U_S'} \\
& - 3(\partial_i \mathrm{U_S'})(\partial^i \mathrm{U_S}) + \mathrm{h}_{ij}'\mathcal{D}^{ij}\mathrm{U_S} + \mathrm{h}_{ij}\mathcal{D}^{ij}\mathrm{U_S'} + 2\mathcal{H}\mathrm{h}_{ij}\mathcal{D}^{ij}\mathrm{U_S} \\
& - 4\pi a^2 \partial^i \mathrm{v}_{\mathrm{P}i}\delta\rho_\mathrm{M} - 2\pi a^2 \, \partial_i(\mathrm{v_N^2}\, \mathrm{v_N^i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})) \\
& - 2\psi\nabla^2\mathrm{U_S'} + 2\mathcal{H}(\partial_i \mathrm{U_S})(\partial^i \mathrm{U_S}) - 4\pi a^2 \mathrm{v}_i(\partial^i \delta\rho_\mathrm{M}) \\
& - 4\pi a^2 \partial^i \big(\mathrm{v}_{\mathrm{N}i}(\delta\rho + \delta p)\big) - 4\pi a^2 \partial^i \big(\mathrm{v}_{\mathrm{N}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\big)\phi \, .
\end{aligned} \tag{D.3}$$

Substituting (D.3) into (D.2), cancelling tensorial terms and simplifying using the lower-order conservation equations, we obtain

$$\begin{aligned}
\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) = & - \delta\rho_\mathrm{eff}' - 3\mathcal{H}(\delta\rho_\mathrm{eff} + \delta p_\mathrm{eff}) + \partial^i \big(\mathrm{v}_{\mathrm{N}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\mathrm{U_S}\big) \\
& - (\bar\rho + \bar p + \delta\rho_\mathrm{S})\partial^i \mathrm{v}_{\mathrm{C}i} - \partial^i \big(\mathrm{v}_{\mathrm{N}i}(\delta\rho + \delta p)\big) + 3(\bar\rho + \bar p + \delta\rho_\mathrm{S})\psi' \\
& + \frac{1}{4\pi a^2}\Big(2\mathcal{H}\mathrm{U_S}\nabla^2\mathrm{U_S} - \mathrm{U_S'}\nabla^2\mathrm{U_S} - 2\mathrm{U_S}\nabla^2\mathrm{U_S'} \\
& \qquad + 2\mathcal{H}(\partial_i \mathrm{U_S})(\partial^i \mathrm{U_S}) - 3(\partial_i \mathrm{U_S'})(\partial^i \mathrm{U_S})\Big) \\
& - \frac{1}{2}\, \partial_i(\mathrm{v_N^2}\, \mathrm{v_N^i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})) - \mathrm{v}_i(\partial^i \delta\rho_\mathrm{M}) - \partial^i \mathrm{v}_{\mathrm{P}i}\delta\rho_\mathrm{M} \\
& - \phi\partial^i \big(\mathrm{v}_{\mathrm{N}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\big) - \partial^i \big(\mathrm{v}_{\mathrm{M}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\big) \\
& .
\end{aligned} \tag{D.4}$$

At this point it is useful to note the following relations, which can be derived without difficulty from lower-order conservation equations:

$$\begin{aligned}
\partial^i \big(\mathrm{v}_{\mathrm{N}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\mathrm{U_S}\big) = & \; (\partial^i \mathrm{U_S})\mathrm{v}_{\mathrm{N}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S}) \\
& - \frac{1}{4\pi a^2}(\mathrm{U_S}\nabla^2\mathrm{U_S'} + \mathcal{H}\mathrm{U_S}\nabla^2\mathrm{U_S}) \, ,
\end{aligned} \tag{D.5}$$

$$3\mathrm{U_S'}(\delta\rho_\mathrm{N} + \bar\rho + \bar p) = -\frac{1}{4\pi a^2}(-3\mathrm{U_S'}\nabla^2\mathrm{U_S} - 3\mathrm{U_S'}\mathcal{H}^2 + 3\mathrm{U_S'}\mathcal{H}') \, , \tag{D.6}$$

$$\frac{1}{4\pi a^2}\mathrm{U_S}\nabla^2(\mathrm{U_S'} + \mathcal{H}\mathrm{U_S}) = -\mathrm{U_S}\, \partial^i \big(\mathrm{v}_{\mathrm{N}i}(\bar\rho + \bar p + \delta\rho_\mathrm{S})\big) \, . \tag{D.7}$$

These relations can be exploited to manipulate Eq. (D.4) into the following form:

$$\begin{aligned}
&\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) + \big(\delta\rho'_{\rm eff} + 3\mathcal{H}(\delta\rho_{\rm eff} + \delta p_{\rm eff})\big) - 3(\bar\rho + \bar p + \delta\rho_{\rm S})\psi' \\
= \;& (\partial^i U_{\rm S}) v_{{\rm N}i}(\bar\rho + \bar p + \delta\rho_{\rm S}) - \phi\partial^i\big(v_{{\rm N}i}(\bar\rho + \bar p + \delta\rho_{\rm S})\big) - \partial^i\big(v_{{\rm M}i}(\bar\rho + \bar p + \delta\rho_{\rm S})\big) \\
&- v_i(\partial^i\delta\rho_{\rm M}) - \partial^i v_{{\rm P}i}\delta\rho_{\rm M} - (\bar\rho + \bar p + \delta\rho_{\rm S})\partial^i v_{{\rm C}i} + 3U'_{\rm S}(\bar\rho + \bar p + \delta\rho_{\rm S}) \\
&- \partial^i\big(v_{{\rm N}i}(\delta\rho + \delta p)\big) - U_{\rm S}\partial^i\big(v_{{\rm N}i}(\bar\rho + \bar p + \delta\rho_{\rm S})\big) - \frac{1}{2}\,\partial_i(v_{\rm N}^2\, v_{\rm N}^i(\bar\rho + \bar p + \delta\rho_{\rm S})) \\
&+ \frac{1}{4\pi a^2}\Big( - 4U'_{\rm S}\nabla^2 U_{\rm S} - 3U'_{\rm S}\mathcal{H}^2 + 3U'_{\rm S}\mathcal{H}' - 4U_{\rm S}\nabla^2 U'_{\rm S} \\
&\quad + 2\mathcal{H}(\partial_i U_{\rm S})(\partial^i U_{\rm S}) - 3(\partial_i U'_{\rm S})(\partial^i U_{\rm S})\Big)\,. \qquad \text{(D.8)}
\end{aligned}$$

Now, by considering the following combination of effective fluid quantities,

$$\begin{aligned}
\delta\rho'_{\rm eff} + 3\mathcal{H}(\delta\rho_{\rm eff} + \delta p_{\rm eff}) &= \big((\bar\rho + \bar p + \delta\rho_{\rm S})v_{\rm N}^2\big)' + 4\mathcal{H}v_{\rm N}^2(\bar\rho + \bar p + \delta\rho_{\rm S}) \\
&+\frac{1}{4\pi a^2}\Big( - 3\mathcal{H}^2 U'_{\rm S} + 3\mathcal{H}' U'_{\rm S} - 4U'_{\rm S}\nabla^2 U_{\rm S} - 4U_{\rm S}\nabla^2 U'_{\rm S} \\
&\quad + 2\mathcal{H}(\partial_i U_{\rm S})(\partial^i U_{\rm S}) - 3(\partial_i U'_{\rm S})(\partial^i U_{\rm S})\Big)\,, \qquad \text{(D.9)}
\end{aligned}$$

it is easy to see that we have obtained precisely Eq. (6.75). This equation has been checked by comparing with the time component of the $\mathcal{O}(\eta^5)$ stress-energy conservation equation.

## D.2. Derivation of the $\mathcal{O}(\eta^5)$ Euler equation

The conformal time derivative of Eq. (6.50) is given by

$$\begin{aligned}
0 = \;& \nabla^2 A'_i + 4\partial_i\big(\psi'' + \mathcal{H}'\phi + \mathcal{H}\phi'\big) \\
&+ 16\pi a^2\Big((\bar\rho + \bar p + \delta\rho_{\rm S})'(v_i - A_i) + (\bar\rho + \bar p + \delta\rho_{\rm S})(v_i - A_i)' + Q_i'^{\;\rm eff}\Big) \\
&+ 16\pi a^2\Big(2\mathcal{H}(\bar\rho + \bar p + \delta\rho_{\rm S})(v_i - A_i) + 2\mathcal{H}Q_i'^{\;\rm eff}\Big) \\
&+ 2A^{j\prime}\partial_i\partial_j U_{\rm S} + 2A^j\partial_i\partial_j U'_{\rm S} + 8\pi a^2\delta\rho'_{\rm S}A_i + 8\pi a^2\delta\rho_{\rm S}A'_i + 16\mathcal{H}\pi a^2\delta\rho'_{\rm S}A_i\,. \qquad \text{(D.10)}
\end{aligned}$$

In order to proceed further, it is now necessary to eliminate the term $\nabla^2 A'_i$. We achieve this by using the divergence of the trace-free part of the $ij$-field equation, $\partial^j G_{\langle ij\rangle} - 8\pi\partial^j T_{\langle ij\rangle} = 0$. Due to the length of the expressions that result, we choose to split the calculation into three sections; the divergence of the trace-free field equation at $\mathcal{O}(\eta^5)$, the $\mathcal{O}(\eta^5)$ gradients of the scalar field equations, and the conservation equation itself.

### D.2.1. Divergence of $\mathcal{O}(\eta^5)$ trace-free field equation

The $\mathcal{O}(\eta^5)$ component of the divergence of the trace-free $ij$ field equation is given by

$$\begin{aligned}
\nabla^2 \mathrm{A}'_i = \; & 16\pi a^2(\mathrm{v}_i - \mathrm{A}_i)\partial^j\big((\bar{\rho}+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}j}\big) + 16\pi a^2\partial^j \mathrm{v}_{\mathrm{P}i}\big((\bar{\rho}+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}j}\big) \\
& + 16\pi a^2(\mathrm{v}_j - \mathrm{A}_j)\partial^j\big((\bar{\rho}+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}i}\big) + 16\pi a^2\partial^j \mathrm{v}_{\mathrm{P}j}\big((\bar{\rho}+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}i}\big) \\
& - \frac{32\pi a^2}{3}(\mathrm{v}^j - \mathrm{A}^j)\partial_i\big((\bar{\rho}+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}j}\big) - \frac{32\pi a^2}{3}\partial_i \mathrm{v}^j_{\mathrm{P}}\big((\bar{\rho}+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})\mathrm{v}_{\mathrm{N}j}\big) \\
& + 8\mathcal{H}\partial_i(\psi' + \mathcal{H}\phi) + 32\mathcal{H}\pi a^2\big((\bar{\rho}+\bar{\mathrm{p}}+\delta\rho_{\mathrm{S}})(\mathrm{v}_i - \mathrm{A}_i) + \mathrm{Q}^{\mathrm{eff}}_i\big) \\
& + \frac{8\mathcal{H}}{3}\mathrm{A}^j\partial_j\partial_i\mathrm{U}_{\mathrm{S}} - \frac{2}{3}\mathrm{A}'^j\partial_j\partial_i\mathrm{U}_{\mathrm{S}} - \frac{2}{3}\mathrm{A}^j\partial_j\partial_i\mathrm{U}'_{\mathrm{S}} - \frac{4}{3}\partial^k\partial^j\mathrm{U}_{\mathrm{S}}\partial_i\mathrm{h}_{kj} - \frac{4}{3}\mathrm{h}^{kj}\partial_k\partial_j\partial_i\mathrm{U}_{\mathrm{M}} \\
& - 16\pi a^2\mathcal{H}\delta\rho_{\mathrm{S}}\mathrm{A}_i - 8\pi a^2\delta\rho_{\mathrm{S}}\mathrm{A}'_i - 8\pi a^2\delta\rho'_{\mathrm{S}}\mathrm{A}_i - \mathrm{A}^j\partial_j\nabla^2\mathrm{B}_i + \frac{4}{3}\mathrm{A}^j\partial_i\nabla^2\mathrm{B}_j \\
& - \frac{28}{3}\partial_j\partial_i\mathrm{U}_{\mathrm{M}}\partial^j\mathrm{U}_{\mathrm{S}} - \frac{28}{3}\partial_j\partial_i\mathrm{U}_{\mathrm{S}}\partial^j\mathrm{U}_{\mathrm{M}} - 4\partial_j\partial_i\mathrm{U}_{\mathrm{S}}\partial^j\phi - \frac{16}{3}\partial_j\partial_i\mathrm{U}_{\mathrm{S}}\partial^j\psi \\
& - \frac{64\pi a^2}{3}\mathrm{U}_{\mathrm{M}}\partial_i\delta\rho_{\mathrm{S}} - \frac{64\pi a^2}{3}\mathrm{U}_{\mathrm{S}}\partial_i\delta\rho_{\mathrm{M}} - \frac{32\pi a^2}{3}\phi\partial_i\delta\rho_{\mathrm{M}} - \frac{32\pi a^2}{3}\psi\partial_i\delta\rho_{\mathrm{M}} \\
& - \frac{16\pi a^2}{3}\delta\rho_{\mathrm{M}}\partial_i\mathrm{U}_{\mathrm{S}} - \frac{16\pi a^2}{3}\delta\rho_{\mathrm{S}}\partial_i\mathrm{U}_{\mathrm{M}} - \frac{32\pi a^2}{3}\delta\rho_{\mathrm{S}}\partial_i\psi + \frac{16\pi a^2}{3}\delta\rho_{\mathrm{S}}\partial_i\phi \\
& + 16\pi a^2\partial^j(\mathrm{v}_{\mathrm{N}i}\mathrm{v}_{\mathrm{N}j}\delta\rho_{\mathrm{M}}) - \frac{16\pi a^2}{3}\partial_i(\mathrm{v}^2_{\mathrm{N}}\delta\rho_{\mathrm{M}}) - \frac{4}{3}\partial_i\nabla^2(\psi + \psi_5 - \phi - \phi_5)\,.
\end{aligned} \tag{D.11}$$

It should be noted that the term $\frac{4}{3}\partial_i\nabla^2(\psi+\psi_5-\phi-\phi_5)$ involves the higher-order potentials $\phi_5 \equiv -\frac{1}{2}(h^{(0,5)} + h^{(1,3)})$ and $\psi_5 \equiv -\frac{1}{2}(h^{(0,5)} + h^{(1,3)})$. These terms do not appear in the scalar field equations at $\mathcal{O}(\eta^4)$, however, they do appear in the $\mathcal{O}(\eta^5)$ field equations, and so cannot be neglected in this calculation. Fortunately, these terms can also be obtained in terms of products of lower-order quantities by calculating gradients of the $\mathcal{O}(\eta^5)$ field equations, enabling us to eliminate them in favour of quantities whose evolution is already known.

### D.2.2. Gradients of the $\mathcal{O}(\eta^5)$ scalar equations

The relevant linear combination of gradients of the $\mathcal{O}(\eta^5)$ scalar field equations that allows us to eliminate the higher-order potentials from the $\mathcal{O}(\eta^5)$ part of the divergence of the trace-free $ij$-field equation is

$$
\begin{aligned}
&\frac{4}{3}\partial_i\nabla^2(\psi+\psi_5-\phi-\phi_5)= \\
&\frac{8\mathcal{H}}{3}\mathrm{A}^j\partial_j\partial_i\mathrm{U_S}+\frac{4}{3}\mathrm{A}'^j\partial_j\partial_i\mathrm{U_S}+\frac{4}{3}\mathrm{A}^j\partial_j\partial_i\mathrm{U}'_\mathrm{S}-\frac{28}{3}\partial_j\partial_i\mathrm{U_M}\partial^j\mathrm{U_S}-\frac{28}{3}\partial_j\partial_i\mathrm{U_S}\partial^j\mathrm{U_M} \\
&-\frac{4}{3}\partial^k\partial^j\mathrm{U_S}\partial_i\mathrm{h}_{kj}-\frac{4}{3}\mathrm{h}^{kj}\partial_k\partial_j\partial_i\mathrm{U_M}-4\partial_j\partial_i\mathrm{U_S}\partial^j\phi-\frac{16}{3}\partial_j\partial_i\mathrm{U_S}\partial^j\psi-16\pi a^2\partial_i\delta\mathrm{p} \\
&-\frac{16\pi a^2}{3}(\mathrm{A}^j+2\mathrm{v}^j)\partial_i\big((\bar\rho+\bar{\mathrm{p}}+\delta\rho_\mathrm{S})\mathrm{v}_{\mathrm{N}j}\big)-\frac{32\pi a^2}{3}\partial_i\mathrm{v}_\mathrm{P}^j\big((\bar\rho+\bar{\mathrm{p}}+\delta\rho_\mathrm{S})\mathrm{v}_{\mathrm{N}j}\big) \\
&+\frac{1}{3}\mathrm{A}^j\partial_i\nabla^2\mathrm{B}_j-16\pi a^2\delta\rho_\mathrm{M}\partial_i\mathrm{U_S}-16\pi a^2\delta\rho_\mathrm{S}\partial_i\mathrm{U_M} \\
&-\frac{64\pi a^2}{3}\mathrm{U_M}\partial_i\delta\rho_\mathrm{S}-\frac{64\pi a^2}{3}\mathrm{U_S}\partial_i\delta\rho_\mathrm{M}-\frac{16\pi a^2}{3}\partial_i(\mathrm{v}_\mathrm{N}^2\delta\rho_\mathrm{M}) \\
&-\frac{32\pi a^2}{3}\phi\partial_i\delta\rho_\mathrm{M}-\frac{32\pi a^2}{3}\psi\partial_i\delta\rho_\mathrm{M}-\frac{16\pi a^2}{3}\delta\rho_\mathrm{M}\partial_i\mathrm{U_S} \\
&-\frac{16\pi a^2}{3}\delta\rho_\mathrm{S}\partial_i\mathrm{U_M}-\frac{32\pi a^2}{3}\delta\rho_\mathrm{S}\partial_i\psi-\frac{32\pi a^2}{3}\delta\rho_\mathrm{S}\partial_i\phi \\
&+4\partial_i(\mathrm{U}''_\mathrm{M}+3\mathcal{H}\mathrm{U}'_\mathrm{M}+2\mathcal{H}^2\mathrm{U_M}+\mathcal{H}'\mathrm{U_M})-16\pi a^2(\bar\rho+\bar{\mathrm{p}})\partial_i\mathrm{U_M} \\
&+4\partial_i(\psi''+2\mathcal{H}\psi'+\mathcal{H}^2\psi+\frac{1}{2}\mathcal{H}'\psi)-8\pi a^2(\bar\rho-\bar{\mathrm{p}})\partial_i\psi-2\Lambda a^2\partial_i\psi \\
&+\partial_i(4\mathcal{H}\phi'+2\mathcal{H}'\phi+4\mathcal{H}^2\phi)-8\pi a^2(\bar\rho+3\bar{\mathrm{p}})\partial_i\phi+2\Lambda a^2\partial_i\phi\ ,
\end{aligned}
\tag{D.12}
$$

We now have all the results we need to demonstrate the conservation of the vector constraint equation.

### D.2.3. Vector conservation equation at $\mathcal{O}(\eta^5)$

Using the result above, we can use Eq. (D.11) to substitute in for $\nabla^2\mathrm{A}'_i$ in Eq. (D.10). This gives

$$
\begin{aligned}
0 = {} & 16\pi a^2\Big((\bar{\rho}+\bar{p}+\delta\rho_S)'(v_i-A_i)+(\bar{\rho}+\bar{p}+\delta\rho_S)(v_i-A_i)'+Q_i'^{\,\mathrm{eff}} \\
&+4\mathcal{H}(\bar{\rho}+\bar{p}+\delta\rho_S)(v_i-A_i)+4\mathcal{H}Q_i^{\mathrm{eff}} \\
&+(v_i-A_i)\partial^j\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Nj}\big)+\partial^j v_{Pi}\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Nj}\big) \\
&+(v_j-A_j)\partial^j\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Ni}\big) \\
&+\partial^j v_{Pj}\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Ni}\big)\Big)+16\pi a^2\partial^j(v_{Ni}v_{Nj}\delta\rho_M)-\frac{32\pi a^2}{3}\partial_i v_P^j\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Nj}\big) \\
&-\frac{32\pi a^2}{3}(v^j-A^j)\partial_i\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Nj}\big)-\frac{16\pi a^2}{3}\partial_i(v_N^2\delta\rho_M)-A^j\partial_j\nabla^2 B_i \\
&+\frac{4}{3}A^j\partial_i\nabla^2 B_j+\frac{8\mathcal{H}}{3}A^j\partial_j\partial_i U_S \\
&+\frac{4}{3}A'^j\partial_j\partial_i U_S+\frac{4}{3}A^j\partial_j\partial_i U_S'-\frac{4}{3}\partial^k\partial^j U_S\partial_i h_{kj}-\frac{4}{3}h^{kj}\partial_k\partial_j\partial_i U_M \\
&-\frac{28}{3}\partial_j\partial_i U_M\partial^j U_S-\frac{28}{3}\partial_j\partial_i U_S\partial^j U_M \\
&-\frac{64\pi a^2}{3}U_M\partial_i\delta\rho_S-\frac{64\pi a^2}{3}U_S\partial_i\delta\rho_M-\frac{32\pi a^2}{3}\phi\partial_i\delta\rho_M-\frac{32\pi a^2}{3}\psi\partial_i\delta\rho_M \\
&-\frac{16\pi a^2}{3}\delta\rho_M\partial_i U_S-\frac{16\pi a^2}{3}\delta\rho_S\partial_i U_M \\
&-\frac{32\pi a^2}{3}\delta\rho_S\partial_i\psi+\frac{16\pi a^2}{3}\delta\rho_S\partial_i\phi+4\partial_i\psi''+8\mathcal{H}\partial_i\psi' \\
&+4\mathcal{H}'\partial_i\psi'+8\mathcal{H}^2\partial_i\phi-\frac{4}{3}\partial_i\nabla^2(\psi+\psi_5-\phi-\phi_5)\,. \qquad \text{(D.13)}
\end{aligned}
$$

We can now substitute in for the gradients of the $\mathcal{O}(\eta^5)$ field equation, $\frac{4}{3}\partial_i\nabla^2(\psi+\psi_5-\phi-\phi_5)$, using Eq. (D.12). After some manipulation we then obtain

$$
\begin{aligned}
0 = {} & 16\pi a^2\Big((\bar{\rho}+\bar{p}+\delta\rho_S)'(v_i-A_i)+(\bar{\rho}+\bar{p}+\delta\rho_S)(v_i-A_i)'+Q_i'^{\,\mathrm{eff}} \\
&+4\mathcal{H}(\bar{\rho}+\bar{p}+\delta\rho_S)(v_i-A_i)+4\mathcal{H}Q_i^{\mathrm{eff}} \\
&+\partial^j(v_{Ni}v_{Nj}\delta\rho_M)+(v_i-A_i)\partial^j\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Nj}\big) \\
&+(v_j-A_j)\partial^j\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Ni}\big)+A^j\partial_i\big((\bar{\rho}+\bar{p}+\delta\rho_S)v_{Nj}\big) \\
&+(\partial^j v_{Pi})v_{Nj}(\bar{\rho}+\bar{p}+\delta\rho_S)+(\partial^j v_{Pj})v_{Ni}(\bar{\rho}+\bar{p}+\delta\rho_S)\Big) \\
&-A^j\partial_j\nabla^2 B_i+A^j\partial_i\nabla^2 B_j+16\pi a^2\partial_i\delta p \\
&+16\pi a^2\big((\bar{\rho}+\bar{p}+\delta\rho_S)\partial_i U_M+\delta\rho_M\partial_i U_S+\delta\rho_S\partial_i\phi\big) \\
&+2\mathcal{H}'\partial_i\phi+4\mathcal{H}^2\partial_i\phi+8\pi a^2(\bar{\rho}+3\bar{p})\partial_i\phi-2\Lambda a^2\partial_i\phi \\
&-2\mathcal{H}'\partial_i\psi-4\mathcal{H}^2\partial_i\psi+8\pi a^2(\bar{\rho}-\bar{p})\partial_i\psi+2\Lambda a^2\partial_i\psi \\
&-4\partial_i U_M''-12\mathcal{H}\partial_i U_M'-4\mathcal{H}'\partial_i U_M-8\mathcal{H}^2\partial_i U_M\,. \qquad \text{(D.14)}
\end{aligned}
$$

Now using the following three relations:

$$\begin{aligned}
\mathrm{Q}_i'^{\,\mathrm{eff}} + 4\mathcal{H}\mathrm{Q}_i^{\mathrm{eff}} &= \delta\rho_{\mathrm{M}}'\mathrm{v}_{\mathrm{N}i} + \delta\rho_{\mathrm{M}}\mathrm{v}_{\mathrm{N}i}' + 4\mathcal{H}\delta\rho_{\mathrm{M}}\mathrm{v}_{\mathrm{N}i} \\
&\quad + \frac{1}{4\pi a^2}\partial_i\big(\mathrm{U}_{\mathrm{M}}'' + 3\mathcal{H}'\mathrm{U}_{\mathrm{M}}' + (2\mathcal{H}^2 + \mathcal{H}')\mathrm{U}_{\mathrm{M}}\big) && \text{(D.15)} \\
2\mathrm{A}^j\partial_{[i}\nabla^2\mathrm{B}_{j]} &= 16\pi a^2\mathrm{A}^j\Big(\partial_j\big((\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}\big)\mathrm{v}_{\mathrm{N}i} - \partial_i\big((\bar{\rho} + \bar{\mathrm{p}} + \delta\rho_{\mathrm{S}}\big)\mathrm{v}_{\mathrm{N}j}\Big) && \text{(D.16)} \\
2\mathcal{H}^2 + \mathcal{H}' &= 12\pi a^2(\bar{\rho} + \bar{\mathrm{p}}) + \Lambda a^2\ , && \text{(D.17)}
\end{aligned}$$

we can simplify to obtain Eq. (6.76), which we confirm can also be directly derived from the stress-energy conservation equations. This calculation demonstrates that Eq. (6.50) is maintained under evolution, as required.

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